NACA TN 3288

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 3288

ON THE ANALYSIS OF LINEAR AND NONLINEAR DYNAMICAL SYSTEMS FROM TRANSIENT-RESPONSE DATA

By Marvin Shinbrot

Ames Aeronautical Laboratory
Moffett Field, Calif.

Washington
December 1954

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SUMMARY

A general theory of the so-called "equations-of-motion" methods for the analysis of linear dynamical systems is developed first. It is then shown that when viewed from this general point of vantage, all of these linear methods can be extended in a straightforward manner to apply to the analysis of nonlinear systems. In addition, through use of this theory, a new method is derived. It is essentially a variation of the well-known "Fourier transform" method for the analysis of linear systems but possesses certain advantages over previous methods. Application and effectiveness of this method are demonstrated by three examples, two of which are nonlinear - one highly so - and the third being of the fourth order.

INTRODUCTION

It has often been suggested (e.g., in ref. 1) that nonlinearities which are ignored in the classical theory of the equations of motion of an aircraft may be responsible for certain unusual phenomena which have been observed in flights of modern high-speed airplanes and missiles. Consequently, it seems desirable to develop methods for the analysis of such nonlinear systems - methods which allow the calculation from measured transient-response data of the nonlinear stability characteristics as well as the classical linear stability derivatives of the aircraft. Several such methods are described in reference 2, the principal one consisting of a generalization of the so-called "derivative method" which was originally devised for use with linear systems (cf. ref. 3). However, the methods described in reference 2 leave something to be desired from both points of view of accuracy of the results and lengthiness of the calculations. In addition, application of these methods requires, in all but the simplest cases, the previous evaluation by some means of those stability characteristics which are linear. In view of these shortcomings, an attempt has been made in the present study to find simpler, more accurate, and more general procedures. The problem is attacked by first examining several well-known methods for the analysis of simple linear systems and then modifying them as necessary to allow their application to more general systems.

Many methods for the analysis of linear systems have been proposed in the past (see, e.g., refs. 3 and 4). In reference 5, these methods have been classified under two main heads: "equations-of-motion" methods and "response-curve-fitting" methods, the former title including the derivative method and what have been called the Laplace transform and the Fourier transform methods (ref. 3), and the latter consisting of such methods as Prony's (refs. 3 and 6) and the techniques of reference 4. Since the response-curve-fitting methods involve the explicit solution of the equations of motion in terms of the physical parameters of the system at hand, they do not seem suitable for use with nonlinear systems. Hence, we shall be concerned solely with the equations-of-motion methods.

Each of these methods has been considered in the literature as an independent entity; apparently, no attempt has ever been made to subsume all of them under a single general theory. For the purposes of the present study, such a theory would be desirable since it seems reasonable to expect first that when viewed from a more general point of view, a generalization of the methods to nonlinear systems might appear; and second that once such a theory is known, it might be possible to develop new methods, superior in certain respects to the old ones from which the theory sprang.

In accordance with this plan, the paper begins with a short presentation of the three best known of the equations-of-motion methods. These methods are examined from a new point of view which is then shown to lead to the general theory for linear systems. The further extension to nonlinear systems is considered next. Based on the general theory, development of a new method for data analysis follows. Finally, some examples of the application of this recommended method are given.

SYMBOLS

α	angle of attack, radians
$\dot{\alpha}$	$\dfrac{d\alpha}{dt}$
b	$\dfrac{L_\alpha}{mV} - \dfrac{M_q}{I_y} - \dfrac{M_{\dot{\alpha}}}{I_y}$
C_0	$\dfrac{M_\delta}{I_y} + \dfrac{L_\delta}{mV}\dfrac{M_q}{I_y}$
C_1	$-\dfrac{L_\delta}{mV}$
δ	elevator deflection, radians
I_y	pitching moment of inertia, slug-ft^2

k $-\frac{M_\alpha}{I_y} - \frac{L_\alpha}{mV}\frac{M_q}{I_y}$

L linear lift force, lb

L_α $\frac{\partial L}{\partial \alpha}$

L_δ $\frac{\partial L}{\partial \delta}$

$L(\alpha)$ nonlinear lift force, lb

m mass of aircraft, slugs

M linear pitching moment, ft-lb

M_α $\frac{\partial M}{\partial \alpha}$

$M_{\dot{\alpha}}$ $\frac{\partial M}{\partial \dot{\alpha}}$

M_δ $\frac{\partial M}{\partial \delta}$

M_q $\frac{\partial M}{\partial q}$

$M(\alpha)$ nonlinear pitching moment, ft-lb

q pitching velocity, radians/sec

t time, sec

V velocity of aircraft, ft/sec

Other symbols will be defined as they are introduced.

ANALYSIS

Some Equations-of-Motion Methods for Linear Systems

In the presentation of the general theory of equations-of-motion methods, it is desirable to have some examples of these methods set down for examination. The three best known of these methods are briefly described below; for a more detailed discussion of them, see reference 3.

As a concrete example, let us consider an airplane operating under conditions where the stability characteristics are effectively linear, so that, as in reference 3, the equations of its longitudinal motion can be written

$$-\alpha L_{\alpha} - \dot{\alpha} mV + qmV = \delta L_{\delta}$$

$$-\alpha M_{\alpha} - \dot{\alpha} M_{\dot{\alpha}} - qM_q + \dot{q} I_Y = \delta M_{\delta}$$

It will be convenient to eliminate q and so to write these equations as

$$\ddot{\alpha}(t) + b\dot{\alpha}(t) + k\alpha(t) = C_o\delta(t) + C_1\dot{\delta}(t) \tag{1}$$

where dots denote differentiation with respect to time t. It is assumed that time histories of $\alpha(t)$ and $\delta(t)$ are available from flight records and that one wishes to calculate the constants b, k, C_o, and C_1. For simplicity, it will further be assumed that the constant b is positive and that $\delta(t)$ is a pulse, so that $\delta(t)$ is zero for t sufficiently large. These restrictions will be removed farther on.

<u>The derivative method</u>.- In order to apply the derivative method, it is necessary first to differentiate the given records to obtain the derivatives $\dot{\alpha}(t)$, $\ddot{\alpha}(t)$, and $\dot{\delta}(t)$. Then, for fixed t, equation (1) may be considered as an equation for the constants b, k, C_o, and C_1. By letting t vary, a set of such equations can be obtained, which can in turn be solved by least squares (see reference 6, page 210, where, however, the integral rather than the sum of the squares of the errors has been minimized) for the desired constants. By this procedure, the following equations are obtained:

$$\left.\begin{aligned}
b\int_0^\infty \dot{\alpha}^2 dt + k\int_0^\infty \dot{\alpha}\alpha dt - C_o\int_0^\infty \dot{\alpha}\delta dt - C_1\int_0^\infty \dot{\alpha}\,\dot{\delta} dt &= -\int_0^\infty \dot{\alpha}\,\ddot{\alpha} dt \\
b\int_0^\infty \alpha\dot{\alpha} dt + k\int_0^\infty \alpha^2 dt - C_o\int_0^\infty \alpha\delta dt - C_1\int_0^\infty \alpha\dot{\delta} dt &= -\int_0^\infty \alpha\ddot{\alpha} dt \\
-b\int_0^\infty \delta\dot{\alpha} dt - k\int_0^\infty \delta\alpha dt + C_o\int_0^\infty \delta^2 dt + C_1\int_0^\infty \delta\dot{\delta} dt &= \int_0^\infty \delta\ddot{\alpha} dt \\
-b\int_0^\infty \dot{\delta}\,\dot{\alpha} dt - k\int_0^\infty \dot{\delta}\alpha dt + C_o\int_0^\infty \dot{\delta}\delta dt + C_1\int_0^\infty \dot{\delta}^2 dt &= \int_0^\infty \dot{\delta}\,\ddot{\alpha} dt
\end{aligned}\right\} \tag{2}$$

Equations (2) can be solved for the desired parameters b, k, C_o, and C_1.

The Laplace transform method.- Letting $A(p)$ and $\Delta(p)$ denote the Laplace transforms of $\alpha(t)$ and $\delta(t)$, respectively, so that

$$\left.\begin{aligned} A(p) &= \int_0^\infty e^{-pt}\,\alpha(t)dt \\ \Delta(p) &= \int_0^\infty e^{-pt}\,\delta(t)dt \end{aligned}\right\} \tag{3}$$

it follows that if $\alpha(t)$ and $\delta(t)$ are related by equation (1), then

$$(p^2 + bp + k)\,A(p) = (C_1 p + C_o)\,\Delta(p) \tag{4}$$

(ref. 7). In writing down equation (4), it has been assumed for simplicity that $\alpha(0) = \dot{\alpha}(0) = \delta(0) = 0$; this restriction is inessential and will be removed later on. For any value of p, equation (4) is an equation in b, k, C_o, and C_1. After finding the Laplace transforms of $\alpha(t)$ and $\delta(t)$ for several such values of p, say for $p = p_1, p_2, \ldots, p_N$, the corresponding equations (4) can be set up and solved by least squares to obtain

$$\left.\begin{aligned}
&b\sum p_i^2A^2(p_i) + k\sum p_iA^2(p_i) - C_o\sum p_iA(p_i)\,\Delta(p_i) - \\
&\qquad C_1\sum p_i^2A(p_i)\,\Delta(p_i) = -\sum p_i^3A^2(p_i) \\
&b\sum p_iA^2(p_i) + k\sum A^2(p_i) - C_o\sum A(p_i)\,\Delta(p_i) - \\
&\qquad C_1\sum p_iA(p_i)\,\Delta(p_i) = -\sum p_i^2A^2(p_i) \\
&-b\sum p_iA(p_i)\,\Delta(p_i) - k\sum A(p_i)\,\Delta(p_i) + C_o\sum \Delta^2(p_i) + \\
&\qquad C_1\sum p_i\Delta^2(p_i) = \sum p_i^2A(p_i)\,\Delta(p_i) \\
&-b\sum p_i^2A(p_i)\,\Delta(p_i) - k\sum p_iA(p_i)\,\Delta(p_i) + C_o\sum p_i\Delta^2(p_i) + \\
&\qquad C_1\sum p_i^2\Delta^2(p_i) = \sum p_i^3A(p_i)\,\Delta(p_i)
\end{aligned}\right\} \tag{5}$$

where all sums are over the range $1, \ldots, N$ of the index i. These equations then can be solved for the desired constants.

The Fourier transform method.- The Fourier transform method proceeds in much the same way as did the Laplace transform method.

Defining

$$\left.\begin{aligned} A(i\omega) &= \int_0^\infty e^{-i\omega t}\,\alpha(t)\,dt \\ \Delta(i\omega) &= \int_0^\infty e^{-i\omega t}\,\delta(t)\,dt \end{aligned}\right\} \tag{6}$$

$(i^2 = -1)$, it follows that if $\alpha(0) = \dot{\alpha}(0) = \delta(0)$, then

$$\left[(i\omega)^2 + i\omega b + k\right] A(i\omega) = \left[C_0 + i\omega C_1\right] \Delta(i\omega) \tag{7}$$

This equation can be written as two real equations by setting

$$\left.\begin{aligned} A(i\omega) &= C(\omega) - iS(\omega) \\ \Delta(i\omega) &= \Gamma(\omega) - i\,\Sigma(\omega) \end{aligned}\right\} \tag{8}$$

Putting equations (6) and (8) together, we see this means that

$$C(\omega) = \int_0^\infty \alpha(t)\cos\omega t\,dt \qquad S(\omega) = \int_0^\infty \alpha(t)\sin\omega t\,dt$$

$$\Gamma(\omega) = \int_0^\infty \delta(t)\cos\omega t\,dt \qquad \Sigma(\omega) = \int_0^\infty \delta(t)\sin\omega t\,dt$$

Breaking equation (7) into its real and imaginary parts gives

$$\omega\,S(\omega)b + C(\omega)k - \Gamma(\omega)C_0 - \omega\,\Sigma(\omega)\,C_1 = \omega^2 C(\omega)$$

$$\omega\,C(\omega)b - S(\omega)k + \Sigma(\omega)C_0 - \omega\,\Gamma(\omega)C_1 = -\omega^2 S(\omega)$$

After $C(\omega)$, $S(\omega)$, $\Gamma(\omega)$, and $\Sigma(\omega)$ have been evaluated for several values of ω, substitution into these equations yields a number of equations for b, k, C_0, and C_1 which can be solved by least squares. These equations, corresponding to (2) and (5), are exceedingly complicated and will not be reproduced here.

The Common Feature of These Methods and the First Generalization

Each of these methods is usually derived as in the preceding section, through use of a certain specialized concept. Thus, the derivative method is based on the fact that for fixed t, equation (1) is linear in b, k, C_0, and C_1, so that a set of simultaneous linear equations for these parameters can be obtained by varying t. The Laplace and Fourier transform methods stem from the theory of these operators. Thus, if $\alpha(t)$ and $\delta(t)$ are related by the linear differential equation (1), their Laplace transforms are related by equation (4). However, equation (4) is linear in the parameters, and so by writing this equation for several values of p, one can solve the resulting equations by least squares for the desired constants. These derivations tend to obscure the common idea which can be shown to lie behind all the methods. This difficulty can be overcome if these particular derivations are forgotten and if attention is fixed entirely on the formal processes whereby the final least squares equations are obtained. With this in mind, let us reconsider the three methods.

The derivative method.- Equations (2) for the derivative method are formally obtained by

(1) Multiplying equation (1) by the four functions $\dot{\alpha}(t)$, $\alpha(t)$, $\delta(t)$, and $\dot{\delta}(t)$ one at a time and
(2) Integrating the results from zero to infinity.

We should forget for the moment the interpretation of this procedure as the solution of equation (1) by least squares, and simply keep the process of multiplication and integration in mind.

The Laplace transform method.- A similar process can be described for the Laplace transform method. Choosing $N(\geq 4)$ positive numbers p_i,

(1) Equation (1) can be multiplied by the functions $e^{-p_i t}$, $i = 1, \ldots, N$, and
(2) The results can be integrated from zero to infinity.

If the resulting equations are solved by least squares, precisely equations (5) for the determination of the parameters by the Laplace transform method are obtained, provided that in step (2) any integrals which arise involving derivatives of $\alpha(t)$ and $\delta(t)$ are integrated by parts to eliminate these derivatives.

It should be noted that although there appears to be one more step here than there was in applying the derivative method - notably an additional least squares following step (2) - this addition is more apparent than real, since it is necessary to apply a least squares process here merely because N (which is generally greater than four) equations are obtained for the four parameters, while in the derivative method exactly

four such equations were obtained. Thus, the least squares step could be eliminated by choosing $N = 4$ (of course, such a choice is not really practical), or, alternatively, such a step could be added to the derivative method by solving the equations resulting from step (2) in that method by least squares instead of in the ordinary way. Of course, such a step would only be made to make the two methods so far described formally more similar; in practice, it would not be performed.

The Fourier transform method.- Finally,

(1) If equation (1) is multiplied by $\cos \omega t$ and $\sin \omega t$ for several values of ω, and
(2) If the results are integrated from zero to infinity (as in the Laplace transform method, integrating by parts to eliminate explicit dependence on the derivatives of α and δ),

one obtains a set of equations identical with those obtained from the Fourier transform method.

The general method for linear systems.- The general development of equations-of-motion methods is now manifest. One takes the equations of motion for the physical system under consideration - for definiteness, say equation (1) - and

(1) Multiplies them by N arbitrary (but sufficiently smooth) functions $y_\nu(t)$.
(2) The resulting equations are then integrated between two definite limits, say, zero and T.

In the three methods described above, $T = \infty$, but this is not essential. In order to avoid some complications initially, we shall continue to integrate over this infinite interval; this restriction will subsequently be removed, however, and T will be allowed to have finite values. In the case of equation (1), the process just described leads to N equations of the form

$$b\int_0^\infty y_\nu(t)\dot{\alpha}(t)dt + k\int_0^\infty y_\nu(t)\alpha(t)dt - C_0\int_0^\infty y_\nu(t)\delta(t)dt - C_1\int_0^\infty y_\nu(t)\dot{\delta}(t)dt = -\int_0^\infty y_\nu(t)\ddot{\alpha}(t)dt, \quad \nu = 1, \ldots, N \tag{9}$$

It is possible that the functions $y_\nu(t)$ depend on $\alpha(t)$ or $\delta(t)$ as, for example, in the derivative method; in such cases, equations (9) can be considered as N equations which are to be solved by least squares for the desired parameters. Of course, this process requires the calculation of the derivatives $\dot{\alpha}(t)$, $\ddot{\alpha}(t)$, and $\dot{\delta}(t)$. On the other hand, if the functions $y_\nu(t)$ are explicitly independent of α and δ, as is the case

in the Laplace and Fourier transform methods, the following formulas, obtained by integrating by parts, are used:

$$\int_0^\infty y_\nu(t)\dot{\alpha}(t)dt = -\ y_\nu(0)\alpha(0) - \int_0^\infty \dot{y}_\nu(t)\alpha(t)dt$$

$$\int_0^\infty y_\nu(t)\ddot{\alpha}(t)dt = -\ y_\nu(0)\dot{\alpha}(0) + \dot{y}_\nu(0)\alpha(0) + \int_0^\infty \ddot{y}_\nu(t)\alpha(t)dt$$

$$\int_0^\infty y_\nu(t)\dot{\delta}(t)dt = -\ y_\nu(0)\delta(0) - \int_0^\infty \dot{y}_\nu(t)\delta(t)dt$$

Substitution into equation (9) gives

$$-\ b\left[y_\nu(0)\alpha(0) + \int_0^\infty \dot{y}_\nu(t)\alpha(t)dt\right] + k\int_0^\infty y_\nu(t)\alpha(t)dt -$$
$$C_o\int_0^\infty y_\nu(t)\delta(t)dt + C_1\left[y_\nu(0)\delta(0) + \int_0^\infty \dot{y}_\nu(t)\delta(t)dt\right] = y_\nu(0)\dot{\alpha}(0) -$$
$$\dot{y}_\nu(0)\alpha(0) - \int_0^\infty \ddot{y}_\nu(t)\alpha(t)dt, \qquad \nu = 1, 2, \ldots, N \qquad (10)$$

Equations (10) are N equations in b, k, C_o, and C_1. If $N \geq 4$, they may be solved by least squares for these parameters.

The choice of the functions $y_\nu(t)$ defines which equations-of-motion method is being used. For this reason, these functions will be referred to as the "method functions."

Generalization to Nonlinear Systems

In this section, all considerations will refer to the equations of longitudinal motion of an aircraft. It will be seen that the method actually is applicable to a far wider class of equations - in particular to the equations of lateral motion of an aircraft, including, if it is so desired, cross-coupling terms. The special analysis presented here can be extended to other problems, the only real restriction being that for practical reasons too many parameters cannot be handled at once.

The following equations, which involve assumptions of constant airspeed, smallness of certain quantities, etc., are often used to describe the motions of an aircraft which has linear stability characteristics:

$$-\ \alpha L_\alpha - \dot{\alpha}mV + qmV = \delta L_\delta$$

$$-\ \alpha M_\alpha - \dot{\alpha}M_{\dot{\alpha}} - qM_q + \dot{q}I_y = \delta M_\delta$$

If, on the other hand, it is assumed that the lift and moment functions are nonlinear in α, these equations become

$$\left.\begin{aligned} -L(\alpha) - \dot{\alpha}mV + qmV &= \delta L_\delta \\ -M(\alpha) - \dot{\alpha}M_{\dot{\alpha}} - qM_q + \dot{q}I_y &= \delta M_\delta \end{aligned}\right\} \tag{11}$$

We shall assume that over the range of interest, approximations of the following form are valid:

$$\left.\begin{aligned} L(\alpha) &= L_1\alpha + L_2\alpha^2 + \ldots + L_l\alpha^l \\ M(\alpha) &= M_1\alpha + M_2\alpha^2 + \ldots + M_n\alpha^n \end{aligned}\right\} \tag{12}$$

where the coefficients L_i, M_i are constant. Only the first three terms of this series will be retained in the present analysis since this three-term approximation usually balances very well the opposing requirements of simplicity and adequate representation of the aerodynamic parameters. If more terms are found to be necessary in a particular problem they can, of course, be added. It should be noted that this three-term approximation is still fairly general, even retaining the possibility of asymmetry in the nonlinearities.

By use of the approximations (12) with $l = n = 3$, equations (11) can be written

$$\left.\begin{aligned} -\frac{L_1}{mV}\alpha - \frac{L_2}{mV}\alpha^2 - \frac{L_3}{mV}\alpha^3 - \dot{\alpha} + q &= \frac{L_\delta}{mV}\delta \\ -\frac{M_1}{I_y}\alpha - \frac{M_2}{I_y}\alpha^2 - \frac{M_3}{I_y}\alpha^3 - \frac{M_{\dot{\alpha}}}{I_y}\dot{\alpha} - \frac{M_q}{I_y}q + \dot{q} &= \frac{M_\delta}{I_y}\delta \end{aligned}\right\} \tag{13}$$

The generalization of the methods of the preceding section to such nonlinear systems proceeds in the obvious way. First, multiply each of equations (13) by N method functions $y_\nu(t)$ which have been selected as suitable. This operation is followed by integration of the resulting 2N equations. If records of the derivatives $\dot{q}$ and $\dot{\alpha}$ are not available, one then eliminates the terms involving these derivatives by integration by parts. The result is N equations in L_1, L_2, L_3, and L_δ and N equations in M_1, M_2, M_3, $M_{\dot{\alpha}}$, and M_δ. If $N \geq 5$, these two sets of equations can be solved by least squares for the parameters.

Of course, the pitching velocity q can be eliminated from equations (11) to yield the single equation

$$\ddot{\alpha} - \left[\frac{M_{\dot{\alpha}}}{I_y} + \frac{M_q}{I_y} - \frac{L'(\alpha)}{mV}\right]\dot{\alpha} - \frac{M(\alpha)}{I_y} - \frac{M_q}{I_y}\frac{L(\alpha)}{mV} = \left(\frac{M_\delta}{I_y} + \frac{M_q}{I_y}\frac{L_\delta}{mV}\right)\delta - \frac{L_\delta}{mV}\dot{\delta}$$

where

$$L'(\alpha) = \frac{d}{d\alpha}L(\alpha)$$

With the approximations (12), this leads to the following generalization of equation (1):

$$\ddot{\alpha} + (b_0 + b_1\alpha + b_2\alpha^2)\dot{\alpha} + (k_0 + k_1\alpha + k_2\alpha^2)\alpha = C_0\delta + C_1\dot{\delta} \tag{14}$$

where

$$\left.\begin{aligned} b_0 &= -\left(\frac{M_{\dot{\alpha}}}{I_y} + \frac{M_q}{I_y} - \frac{L_1}{mV}\right) & k_0 &= -\left(\frac{M_1}{I_y} + \frac{M_q}{I_y}\frac{L_1}{mV}\right) \\ b_1 &= \frac{2L_2}{mV} & k_1 &= -\left(\frac{M_2}{I_y} + \frac{M_q}{I_y}\frac{L_2}{mV}\right) \\ b_2 &= \frac{3L_3}{mV} & k_2 &= -\left(\frac{M_3}{I_y} + \frac{M_q}{I_y}\frac{L_3}{mV}\right) \end{aligned}\right\} \tag{15}$$

By applying the method described to equation (14), the constants b_0, b_1, b_2, k_0, k_1, k_2 can be calculated.

Solution for the parameters in equations of the general form of equation (14) is of interest to workers in many fields. In addition, the evaluation, from these parameters, of the stability constants occurring on the right sides of equations (15) is of considerable interest to aeronautical engineers, and so this problem will be considered in further detail. One cannot, in general, isolate the constants L_i and M_i to obtain, from values of the b_i and the k_i, the nonlinear functions $L(\alpha)$ and $M(\alpha)$. For this reason, it is ordinarily best to apply the method directly to equations (13) rather than to equation (14), provided that records of both $q(t)$ and $\alpha(t)$ are available.

On the other hand, there is one case of interest when equation (14) can be used directly and measurements of $q(t)$ are not needed. It sometimes happens that while the pitching moment $M(\alpha)$ is nonlinear, the lift $L(\alpha)$ can still be successfully approximated by a linear function. In this case, we have the approximations

$$\left.\begin{aligned} L(\alpha) &= L_\alpha\alpha \\ M(\alpha) &= M_1\alpha + M_2\alpha^2 + M_3\alpha^3 \end{aligned}\right\} \tag{16}$$

and so equation (14) becomes

$$\ddot{\alpha} + b\dot{\alpha} + (k_0 + k_1\alpha + k_2\alpha^2)\alpha = C_0\delta + C_1\dot{\delta} \tag{17}$$

where

$$\left.\begin{aligned} k_0 &= -\left(\frac{M_1}{I_y} + \frac{M_q}{I_y}\frac{L_\alpha}{mV}\right) \\ k_1 &= -\frac{M_2}{I_y} \\ k_2 &= -\frac{M_3}{I_y} \end{aligned}\right\} \tag{18}$$

Thus in this case, $M(\alpha)$ can, except for the term $\frac{M_q}{I_y}\frac{L_\alpha}{mV}$ occurring in the expression for k_0, be obtained from an analysis of equation (17). At high speeds, however, this term is small and its effect on the curve of $M(\alpha)$ versus α can be neglected. Thus, the expression for k_0 in formulas (18) can be replaced by the expression

$$k_0 = -\frac{M_1}{I_y} \tag{19}$$

and in this case, the nonlinear moment $M(\alpha)$ is completely determined by the knowledge of k_0, k_1, k_2, and, of course, I_y.

Choice of the Method Functions

Up to this point in the general discussion, the method functions $y_\nu(t)$ have been to a very great extent arbitrary, having to satisfy only certain weak smoothness conditions. In this section, the possiblity of developing new and perhaps improved methods for data analysis by means of a particular choice of the method functions will be explored.

Previous experience, consisting in part of unpublished analyses performed at Ames Aeronautical Laboratory, have indicated that the Fourier transform method generally results in greater accuracy than either the

derivative or the Laplace transform methods.[1] For this reason, the method to be discussed will be a variation of the Fourier transform method.

Let us consider first some evident shortcomings in the Fourier transform method. These defects will offer definite goals to be held in mind in the development of a new method. First, there is a weakness in the Fourier transform method in that all integrations proceed over the interval from zero to infinity (cf. eq. (10)). This causes difficulties in any example in which $\alpha(t)$ and $\delta(t)$ do not approach zero so rapidly that their integrals exist. Thus, referring, for example, to equation (1), if b is not positive or if $\delta(t)$ does not approach zero quickly enough, the method cannot be applied straightforwardly. Furthermore, even if $b > 0$ and $\delta(t) \longrightarrow 0$ in such a way that $\int^{\infty} \alpha(t)dt$ exists, the experimental record often is not long enough for this integral to be accurately calculable when the system is so lightly damped (b small) that sizable oscillations persist even to the end of the run. One device which is sometimes used to overcome this difficulty is equivalent to a change in the method functions. Instead of the functions $\sin \omega t$ and $\cos \omega t$, the functions $e^{-\beta t} \sin \omega t$ and $e^{-\beta t} \cos \omega t$, with some fixed constant β, are used. However, this leads to the same objection that was voiced in footnote 1 for the derivative and Laplace transform methods, notably that the method functions approach zero. Other tricks for dealing with such deficiencies in the Fourier transform method can be evolved, but, rather than develop new devices for each special case, it appears wiser to construct a generally applicable method in which these difficulties never arise - that is, one in which the integration proceeds only over a finite interval.

The second defect which we shall consider becomes clear from an inspection of equation (10), with the functions $y_\nu(t)$ of the form $\sin \omega t$ and $\cos \omega t$ for certain values of ω. Referring to equation (10), it is easily seen that one point, namely the point $t = 0$, is weighted very heavily because of the occurrence of the quantities $\alpha(0)$, $\dot{\alpha}(0)$, and $\delta(0)$. It should be noted that not only are the values of $\alpha(t)$ and $\delta(t)$ at one point relied on to this great extent, but that even the relatively inaccurate value of the derivative of $\alpha(t)$ at that point is weighted. Thus, advantages in accuracy might be expected to accrue if these terms were eliminated.

Since the method for overcoming this second deficiency in the Fourier transform method will also be used in treating the first, the problem of eliminating dependence on the initial values will be discussed now. To this end, consider equation (10). We begin with the Fourier transform

[1]It would seem that no rational explanation for this conclusion has heretofore been offered. However, the theory described herein appears to afford such an explanation. A long and rather tedious analysis based on this theory has indicated that the failure of both the derivative and the Laplace transform methods is due in large part to the fact that the associated method functions approach zero very rapidly as time progresses.

method, that is, with method functions $y_\nu(t)$ of the forms $\sin \omega t$ and $\cos \omega t$. Noting that most of the terms which depend on the initial values at $t = 0$ are multiplied by $y_\nu(0)$, it is seen that a choice of method functions such that $y_\nu(0)$ is zero for all $\nu = 1, 2, \ldots, N$ represents a step in the right direction. Such a choice is easy to make, since it is only necessary to eliminate those method functions which have the form $\cos \omega t$; then, we may write

$$y_\nu(t) = \sin \omega_\nu t, \qquad \nu = 1, 2, \ldots, N \tag{20}$$

This does not entirely eliminate the dependence on the initial conditions, however, as the term $\dot{y}_\nu(0)\alpha(0)$ remains in equation (10). If this term can also be removed, the second weakness in the Fourier transform method will have been entirely corrected. This will clearly be the case if $\dot{y}_\nu(0)$ as well as $y_\nu(0)$ is zero for all $\nu = 1, \ldots, N$. A possible choice of the method functions for which this is so, a choice which still retains the advantages of the favored Fourier transform method, is the following:

$$y_\nu(t) = \sin^2 \omega_\nu t = \frac{1 - \cos 2\omega_\nu t}{2}, \qquad \nu = 1, \ldots, N \tag{21a}$$

With this choice of the method functions, equation (10) becomes

$$\begin{gathered} -b\int_0^\infty \alpha(t)\dot{y}_\nu(t)dt + k\int_0^\infty \alpha(t)y_\nu(t)dt - C_0\int_0^\infty \delta(t)y_\nu(t)dt + \\ C_1\int_0^\infty \delta(t)\dot{y}_\nu(t)dt = -\int_0^\infty \alpha(t)\ddot{y}_\nu(t)dt \end{gathered} \tag{22}$$

The method functions (21a) would be used for systems satisfying differential equations, like (1), which involve derivatives of the second order. More generally, and for the same reasons, if the highest order derivative occurring in any of the equations of motion of a system is the nth, the following method functions are suggested:

$$y_\nu(t) = \sin^n \omega_\nu t, \qquad \nu = 1, \ldots, N \tag{21b}$$

Thus, in the case of the two-degrees-of-freedom system described by equations (13), there is no point in using formula (21a); the simpler method functions (20) may as well be used.

As for the first of the weaknesses in the Fourier transform method, that which is due to integration over an infinite interval, it would

appear at first glance to be easily disposed of by merely choosing some finite, positive number T and integrating over the interval from zero to T. This, however, introduces another difficulty. Suppose $y_\nu(t)$ is given by equation (21a), so that $y_\nu(0) = \dot{y}_\nu(0) = 0$. Returning to the derivation of equation (10), it may be seen that if one integrates only over the interval $0 \leq t \leq T$, all the good which has been achieved by eliminating dependence on the point $t = 0$ is obviated by certain terms which arise - terms of the form $y_\nu(T)\alpha(T)$, $\dot{y}_\nu(T)\alpha(T)$, and $y_\nu(T)\delta(T)$. Thus, the heavy dependence on the initial conditions is replaced by a dependence on the final conditions. Of course, the same approach as was used for eliminating the initial conditions can be used again - that is, the method functions can be chosen in such a way that $y_\nu(T) = \dot{y}_\nu(T) = 0$. One possibility, naturally, is to choose the frequencies ω_ν such that

$$\sin \omega_\nu T = 0, \qquad \nu = 1, \ldots, N$$

Thus, we can set

$$\omega_\nu = \frac{\nu\pi}{T}, \qquad \nu = 1, \ldots, N$$

This choice of the frequencies (corresponding to the method functions given by equation (21b)) leads to an elegant method which gives satisfactory results in certain cases. On the other hand, the difference $\left(\frac{\pi}{T}\right)$ between two successive frequencies is too large to define the "frequency response" (to use loosely the terminology of the Fourier transform) of some examples adequately. For this reason, we should like to be able to choose the frequencies as follows:

$$\omega_\nu = \frac{\nu\pi}{2T}, \qquad \nu = 1, \ldots, N \tag{23}$$

Of course, this means that for frequencies having an odd subscript, $y_\nu(T)$ will be different from zero. To overcome this difficulty, the following choice of the method functions can be made: If the highest derivative occurring in the equations of motion is the nth, define the method functions by the formulas

$$\left.\begin{aligned} y_{2\mu}(t) &= \sin^n \omega_{2\mu} t \\ y_{2\mu+1}(t) &= \begin{cases} \sin^n \omega_{2\mu+1} t, & 0 \leq t \leq \frac{2\mu}{2\mu+1} T \\ 0, & \frac{2\mu}{2\mu+1} T < t \leq T \end{cases} \end{aligned}\right\} \tag{24}$$

where the frequencies ω_ν are given by equation (23) and T is the length of the run. What has been done in choosing these method functions is the following: Those method functions which are such that $y_\nu(T)$ is zero (i.e., those method functions which have even subscripts), have been left unaltered in the form given by equation (21b). The remaining method functions have had the last quarter cycle, during which they would normally have varied from 0 to ±1, chopped off, so that they are identically zero over part of the interval $0 \leq t \leq T$. The claim is not made that this is the best of all possible choices for the method functions. There is certainly no reason for such an assertion, particularly in view of the fact that a certain amount of the data is not used by each of the odd-numbered method functions. However, this amount is small after all and no datum is completely discarded, since the even-numbered method functions use all the data. That these method functions do seem to be adequate is indicated by the results obtained in the examples given below.

Before proceeding to the examples, one further change, imposed in the interest of simplicity in the computations, will be made in the method functions. For the odd-numbered method functions, certain complications arise in the computations due to the fact that the point $(2\mu/2\mu+1)T$ at which the function is cut off may not coincide with a point at which the data are tabulated. For this reason, values of even-numbered frequencies will be chosen in accordance with equation (23). The odd-numbered frequencies, on the other hand, will be changed (by as little as possible, to be sure) in such a way that the following condition is satisfied. Let the data be tabulated at the points $t = t_0, t_1, \ldots, t_K$, where $t_{k+1} - t_k =$ constant. The odd-numbered frequencies are then assumed to be chosen such that

$$\sin \omega_{2\mu+1} t_\alpha = 0$$

where t_α is that one of the two tabulation instants closest to the time $(2\mu/2\mu+1)T$ having an even subscript. The reason for this last condition is merely that it is convenient for a numerical integration procedure (such as the one in the Appendix) to have an even number of intervals (t_k, t_{k+1}); thus, Simpson's rule, for example, calls for an even number of intervals in its application.

For many of the experiments performed on airplanes and missiles, the run is 2 seconds long and the time between tabulated points is 0.05 second, so that the tabulation times t_α are at 0, 0.05, 0.10, . . ., 2.00 seconds. For these values of T and Δt, the rule given above for the frequencies ω_ν is climaxed by the following table:

ν	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
ω_ν	$\frac{\pi}{2}$	$\frac{10\pi}{13}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{30\pi}{17}$	2π	$\frac{20\pi}{9}$	$\frac{5\pi}{2}$	$\frac{25\pi}{9}$	3π	$\frac{10\pi}{3}$	$\frac{7\pi}{2}$	$\frac{70\pi}{19}$	4π

The frequency ω_1 has been omitted from the table since, according to the rule for the determination of the method functions, $y_1(t) \equiv 0$ (see eq. (24), ff.) and offers no information. A 2-second interval will be taken as standard in this report, and all computations will be based on such an interval. If one has a data run T seconds in length, it can of course be brought into this standard form by a preliminary substitution of the form

$$t = \frac{T}{2} \tau$$

and we always assume such a transformation has been made.

Thus, finally, the method may be summarized as follows: Select N frequencies in accordance with the rule given in the preceding paragraph. (The number N is chosen, as in the Fourier transform method, large enough to cover the frequency range of interest in the particular problem; usually, $N = 16$ is adequate.) Multiply the equations of motion by the method functions (24) and integrate the resulting equations from zero to T, where T denotes the length of the data run. Eliminating all explicit dependence of these equations on derivatives of the data by successive integrations by parts results in N linear simultaneous equations for the parameters. The coefficients in these equations are all integrals involving the recorded data; after these have been evaluated by some means,[2] the equations can be solved by least squares for the desired parameters.

EXAMPLES

Three example problems will be solved in order to demonstrate the effectiveness of the proposed analysis method and to illustrate associated computing techniques. An effort was made to select examples representative of problems which often occur in aircraft-response flight testing and which have not been handled adequately by other known analysis methods. These examples have been simplified in some respects, not because of fundamental limitations of the method, but in order to avoid obscuring the essentials of the method and of the related computing techniques.

Although limited use was made of automatic digital computing machinery in the following analyses, it did not appear worthwhile to mechanize complete calculation procedures for these isolated illustrative examples. However, the method appears to be well suited to such mechanization.

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[2]See, for example, the Appendix, where a technique well suited for the type of integrations needed for this method is described.

Example I

The first example concerns the longitudinal response of a hypothetical missile for which it is assumed that the lift varies linearly with angle of attack, although the pitching moment does not. The velocity of the missile is assumed to be sufficiently high for the expression (18) for k_o to be simplified to

$$k_o = -\frac{M_1}{I_y}$$

so that equation (17) can be used to determine the lift and moment characteristics from a transient response. A pulse response of a system described by equation (17) was obtained from a Reeves Electronic Analogue Computer and it was decided to consider this response as given data to be analyzed. The moment of inertia of the missile was chosen to be 100 slug-feet^2. The nonlinear moment curve $M(\alpha)$ versus α which was used to obtain the data is shown as the solid curve in figure 1. The linear stability derivatives were chosen in such a way that the damping parameter is given by

$$b = 2 \tag{25}$$

In order to simplify the presentation, it was decided that free oscillations alone would be analyzed to determine only the constants occurring on the left-hand side of equation (17). Thus, for the data which will be analyzed, $\delta(t) = 0$, and equation (17) becomes

$$\ddot{\alpha} + b\dot{\alpha} + (k_o + k_1\alpha + k_2\alpha^2)\alpha = 0 \tag{26}$$

A plot of the $\alpha(t)$ "data" is given in figure 2, and this information is listed in table I.

Since equation (26) is of the second order, we shall choose, according to the rule given earlier, the method functions (24) with $n = 2$; since the run is 2 seconds long and the time interval between data points is 0.05 second, the frequencies ω_ν are chosen as in the table on page 16.

Integrating factors $\Gamma_n(y_\nu)$ corresponding to each function $y_\nu(t)$ and its first two derivatives are tabulated in columns 6 through 50 of table I. (As discussed in the Appendix, the $\Gamma_n(y_\nu)$ are numbers chosen such that for any integrable function $x(t)$, the sum

$$\sum_n x(t_n)\Gamma_n(y_\nu)$$

is an approximation to the integral of $x(t)y_v(t)$.) Accordingly, the sums displayed beneath table I are the integrals needed for the reduction of the data, divided by the factor $\Delta t = 0.05$. Since this factor occurs homogeneously throughout the equations from which the parameters are to be found (i.e., the generalization of eqs. (22) to the nonlinear eq. at hand), it can be divided out of these equations, and the sums can be used directly without first multiplying them by Δt.

In accordance with the method as it has been described, the equations which are to be solved by least squares have the form

$$\frac{1}{0.05}\left[-b\int_0^2 \alpha(t)\dot{y}_v(t)dt + k_0\int_0^2 \alpha(t)y_v(t)dt + k_1\int_0^2 \alpha^2(t)y_v(t)dt + k_2\int_0^2 \alpha^3(t)y_v(t)dt\right] = -\frac{1}{0.05}\int_0^2 \alpha(t)\ddot{y}_v(t)dt$$

The sums below table I are needed for the evaluation of the coefficients of b, k_0, k_1, and k_2 in the above equation. These sums are again listed in table II, and the coefficients in the last equation are set down as columns 4, 5, 6, 7, and 9 of table II. The sums displayed beneath table II are needed for the final least squares step of the solution. Using these sums, it can be seen that the following equations are to be solved for the parameters:

$$\left.\begin{aligned}
24.1669\,b - 1.05270\,k_0 - 0.336760\,k_1 - 0.0140294\,k_2 &= -7.01790\\
-1.05270\,b + 0.885340\,k_0 + 0.144127\,k_1 + 0.00906192\,k_2 &= 45.5226\\
-0.336760\,b + 0.144127\,k_0 + 0.0374138\,k_1 + 0.00189656\,k_2 &= 7.00244\\
-0.0140294\,b + 0.00906192\,k_0 + 0.00189656\,k_1 + 0.000108975\,k_2 &= 0.459829
\end{aligned}\right\} \tag{27}$$

It may seem odd to some that four significant figures have been used in table II for the values of the integrals and six significant figures for the coefficients in equations (27), while the test data are not given to more than three significant figures. The reason for carrying more significant figures in the computations than there are in the data is to avoid eventual loss of accuracy due to round-off errors and other errors of a similar type. This procedure of carrying a few more (fictitious) figures than the data supply is usually necessary in order to retain even the basic information which is in the data.

Solving equations (27) gives

$$\left.\begin{aligned} b &= 1.95 \\ k_o &= 50.4 \\ k_1 &= -\ 30.5 \\ k_2 &= 806 \end{aligned}\right\} \tag{28}$$

The only parameter whose numerical value is given and which can be immediately checked is the damping parameter. Comparing the values of b given in equations (25) and (28), we see that it has been found with an error of 2.5 percent. The constants k_o, k_1, k_2 cannot be checked directly; however, the calculated pitching-moment curve

$$\begin{aligned} M(\alpha) &= -\ I_y(k_o\alpha + k_1\alpha^2 + k_2\alpha^3) \\ &= -\ 5040\ \alpha + 3050\ \alpha^2 - 80600\ \alpha^3 \end{aligned}$$

can be plotted and compared with the true curve from which we started. This has been done in figure 1 from which it can be seen that the error at the least accurate point is less than 3 percent.

It should be noted that the values of $\Gamma_n(y_\nu)$ given in table I can be used to solve any problem of the type considered here which depends on a second-order differential equation or on a system of such equations. If the data run is 2 seconds long, it is only necessary to insert the data in table I in place of the data used in this example and proceed as we have just done. As mentioned earlier, if the data run is more or less than 2 seconds long, it is only necessary to make a preliminary transformation of the time scale so that in the new time scale the data run is 2 seconds long, a process illustrated in example II.

Example II

The first example served to illustrate the application of the method to an equation of the form (14), corresponding in the missile pitch-response problem to the case where only $\alpha(t)$ and $\delta(t)$ are measured. A problem involving equations, like (13), of the first order, corresponding to the case where $q(t)$ is available in addition to $\alpha(t)$ and $\delta(t)$, will be illustrated now.

As in example I, the lift force will be assumed linear and the pitching moment nonlinear. The following parametric values were assumed:

$$\left.\begin{array}{ll} m = 2 & L_\alpha = 1000 \\ V = 750 & M_{\dot{\alpha}} = -200 \\ I_y = 100 & M_q = -500 \end{array}\right\} \tag{29}$$

The nonlinear $M(\alpha)$ is plotted as the solid curve in figure 3. It should be noted that in contrast to the first example, the pitching moment is unstable at $\alpha = 0$ and highly nonlinear.

The "test data" were manufactured by determining a pulse response of this missile on the REAC. Again, the control characteristics of the system will not be considered, so that only free oscillations are shown in figure 4.

Merely to have a standard length of run, a 2-second interval was always selected for the calculation of the integrating factors $\Gamma_n(y_\nu)$ (see the Appendix for the definition of these quantities). To illustrate the computation procedure for data runs of different lengths, a 1-second run will be considered in the present example.

In order to use the integrating factors displayed in table III, it is necessary to make a preliminary transformation of the form

$$t = \frac{\tau}{2}$$

This transformation has the following effect on equations (13):

$$\left.\begin{array}{r} -\dfrac{L_1}{mV}\alpha - \dfrac{L_2}{mV}\alpha^2 - \dfrac{L_3}{mV}\alpha^3 - 2\dfrac{d\alpha}{d\tau} + q = \dfrac{L_\delta}{mV}\delta \\ -\dfrac{M_1}{I_y}\alpha - \dfrac{M_2}{I_y}\alpha^2 - \dfrac{M_3}{I_y}\alpha^3 - 2\dfrac{M_{\dot{\alpha}}}{I_y}\dfrac{d\alpha}{d\tau} - \dfrac{M_q}{I_y}q + 2\dfrac{dq}{d\tau} = \dfrac{M_\delta}{I_y}\delta \end{array}\right\} \tag{30}$$

Recalling that for the problem under discussion, $L_1 = L_\alpha$, $L_2 = L_3 = 0$, $\delta(t) = 0$, it can be seen from equations (30) that the equations to be solved by least squares for the parameters have the forms

$$-\frac{L_\alpha}{mV}\int_0^2 \alpha(\tau)y_\nu(\tau)d\tau + 2\int_0^2 \alpha(\tau)\frac{dy_\nu(\tau)}{d\tau}d\tau + \int_0^2 q(\tau)y_\nu(\tau)d\tau = 0$$

and

$$-\frac{M_1}{I_y}\int_0^2 \alpha(\tau)y_\nu(\tau)d\tau - \frac{M_2}{I_y}\int_0^2 \alpha^2(\tau)y_\nu(\tau)d\tau - \frac{M_3}{I_y}\int_0^2 \alpha^3(\tau)y_\nu(\tau)d\tau +$$

$$2\frac{M_{\dot{\alpha}}}{I_y}\int_0^2 \alpha(\tau)\frac{dy_\nu(\tau)}{d\tau}d\tau - \frac{M_q}{I_y}\int_0^2 q(\tau)y_\nu(\tau)d\tau - 2\int_0^2 q(\tau)\frac{dy_\nu(\tau)}{d\tau}d\tau = 0$$

The "data" of figure 4 are presented as functions of τ in table III. The sums, which when multiplied by $\Delta\tau = 0.05$ approximate the integrals in the last two equations, are given below table III. These sums have been listed in the appropriate places in table IV. With circled numbers referring to columns in table IV, it can be seen that the above two equations are equivalent to the following:

$$-\frac{L_\alpha}{mV}③ + 2⑧ + ② = 0$$

$$-\frac{M_1}{I_y}③ - \frac{M_2}{I_y}④ - \frac{M_3}{I_y}⑤ + 2\frac{M_{\dot{\alpha}}}{I_y}⑧ - \frac{M_q}{I_y}② - 2⑨ = 0$$

Hence,

$$L_\alpha = mV\frac{2\Sigma ③ \times ⑧ + \Sigma ② \times ③}{\Sigma ③^2}$$

or

$$L_\alpha = 1033$$

using the given values of m and V (eqs. (29)), while the equations for the moment parameters are

$$\frac{M_1}{I_y}\sum \textcircled{3}^2 + \frac{M_2}{I_y}\sum \textcircled{3}\times\textcircled{4} + \frac{M_3}{I_y}\sum \textcircled{3}\times\textcircled{5} - \left(2\,\frac{M_{\dot{\alpha}}}{I_y}\right)\sum \textcircled{3}\times\textcircled{8} + \frac{M_q}{I_y}\sum \textcircled{2}\times\textcircled{3} = -2\sum \textcircled{3}\times\textcircled{9}$$

$$\frac{M_1}{I_y}\sum \textcircled{3}\times\textcircled{4} + \frac{M_2}{I_y}\sum \textcircled{4}^2 + \frac{M_3}{I_y}\sum \textcircled{4}\times\textcircled{5} - \left(2\,\frac{M_{\dot{\alpha}}}{I_y}\right)\sum \textcircled{4}\times\textcircled{8} + \frac{M_q}{I_y}\sum \textcircled{2}\times\textcircled{4} = -2\sum \textcircled{4}\times\textcircled{9}$$

$$\frac{M_1}{I_y}\sum \textcircled{3}\times\textcircled{5} + \frac{M_2}{I_y}\sum \textcircled{4}\times\textcircled{5} + \frac{M_3}{I_y}\sum \textcircled{5}^2 - \left(2\,\frac{M_{\dot{\alpha}}}{I_y}\right)\sum \textcircled{5}\times\textcircled{8} + \frac{M_q}{I_y}\sum \textcircled{2}\times\textcircled{5} = -2\sum \textcircled{5}\times\textcircled{9}$$

$$-\frac{M_1}{I_y}\sum \textcircled{3}\times\textcircled{8} - \frac{M_2}{I_y}\sum \textcircled{4}\times\textcircled{8} - \frac{M_3}{I_y}\sum \textcircled{5}\times\textcircled{8} + \left(2\,\frac{M_{\dot{\alpha}}}{I_y}\right)\sum \textcircled{8}^2 - \frac{M_q}{I_y}\sum \textcircled{2}\times\textcircled{8} = 2\sum \textcircled{8}\times\textcircled{9}$$

$$\frac{M_1}{I_y}\sum \textcircled{2}\times\textcircled{3} + \frac{M_2}{I_y}\sum \textcircled{2}\times\textcircled{4} + \frac{M_3}{I_y}\sum \textcircled{2}\times\textcircled{5} - \left(2\,\frac{M_{\dot{\alpha}}}{I_y}\right)\sum \textcircled{2}\times\textcircled{8} + \frac{M_q}{I_y}\sum \textcircled{2}^2 = -2\sum \textcircled{2}\times\textcircled{9}$$

When the sums displayed below table IV are inserted in the appropriate places, the resulting equations can be solved to yield

$$\frac{M_1}{I_y} = 521 \qquad 2\frac{M_{\dot{\alpha}}}{I_y} = 38.66$$

$$\frac{M_2}{I_y} = -42.3 \qquad \frac{M_q}{I_y} = -26.7$$

$$\frac{M_3}{I_y} = -197,000$$

Hence, using the value of I_y given in equations (29),

$$\left.\begin{aligned} M_1 &= 5.21\times10^4 & M_{\dot{\alpha}} &= 1930 \\ M_2 &= -4.23\times10^3 & M_q &= -2670 \\ M_3 &= -1.97\times10^7 & & \end{aligned}\right\} \tag{31}$$

To begin our discussion of these values, consider L_α. A comparison of the values given for L_α by equations (29) and (31) shows that L_α has been found within 3.2 percent. The nonlinear pitching moment

$$M(\alpha) = (5.21\times10^4)\alpha - (4.23\times10^3)\alpha^2 - (1.97\times10^7)\alpha^3$$

has been superposed as the dotted curve on the true moment curve in figure 3. As can be seen, the agreement for this strongly nonlinear problem is excellent. Finally, the errors in the calculated values of the parameters $M_{\dot{\alpha}}$ and M_q are enormous, but the reason for this is well known and easily explained. Consider equations (13) which describe the motion. Eliminating q from these equations results in equation (14); since the lift is linear, $b_1 = b_2 = 0$. The constant b_0, on which M_q and $M_{\dot{\alpha}}$ have their principal effect, is easily interpreted physically as a measure of the damping in the system. The other gross aspects of the response are relatively little affected by either M_q or $M_{\dot{\alpha}}$. However, the important quantity in b_0 is not M_q or $M_{\dot{\alpha}}$ alone, but is their sum, $M_q + M_{\dot{\alpha}}$. In other words, relatively large changes in M_q and $M_{\dot{\alpha}}$ are possible without causing any great change in the motion, just so long as their sum remains constant. Thus, one may not expect to find M_q and $M_{\dot{\alpha}}$ accurately from an experiment such as this - only their sum may be relied upon. This is verified in the present example, since equations (29) give the sum

$$M_q + M_{\dot{\alpha}} = -700$$

while equations (31) give

$$M_q + M_{\dot{\alpha}} = -740$$

These two values differ by only 5.7 percent.

It should be noted that the assumption, which has been made in this and the preceding example, of the linearity of the lift is not necessary. If it is suspected that the lift is linear but if no definite verification of this is available, a nonlinear form such as (12) can be assumed for the lift, and the coefficients $L_1, L_2, L_3, \ldots$, can be calculated. If the lift is indeed linear, it should turn out that $L_2, L_3, \ldots$, are small. Some limited experience has shown that this method does work but that the errors in the calculated parameters are somewhat larger than when the correct form is assumed. The reason for this is not known, but it appears that it may be associated with a tendency of the extraneous parameters L_2 and L_3 to fit the lift curve to that corresponding to the original data, errors and all, at the expense of the smoothing operation which is necessary with this type of data and which is performed by the least squares process when the correct form is assumed.

Example III

We turn, finally, to a system described by a differential equation whose order is higher than the second. Since the higher order systems whose occurrence is most common appear to be those of the fourth order, we shall be concerned with such a system. In order to simplify the presentation, a linear system will be considered; there are no conceptual difficulties in the generalization to the nonlinear case. In addition, it will be assumed for simplicity that free oscillations are available for analysis. Thus, the system to be analyzed is assumed to be described by an equation of the form

$$\frac{d^4x}{dt^4} + a_3 \frac{d^3x}{dt^3} + a_2 \frac{d^2x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x = 0 \qquad (32)$$

A solution of this equation was calculated over the interval from 0 to 2 seconds, for the following values of the coefficients a_i:

$a_0 = 2544.9$ $\qquad$ $a_1 = 219.32$

$a_2 = 132.87$ $\qquad$ $a_3 = 2.000$

The result, representing the free oscillations in response to some disturbance, is tabulated in table V and presented graphically as the solid curve in figure 5. The sums needed for the solution of the problem are displayed below table V and again in table VI. The least-squares equations for the parameters are

$$\left.\begin{aligned}
24.2035\, a_0 - 16.7225\, a_1 - 1098.19\, a_2 + 1603.19\, a_3 &= -81036.4 \\
-16.7225\, a_0 + 855.538\, a_1 + 664.495\, a_2 - 82282.3\, a_3 &= 85777.1 \\
-1098.19\, a_0 + 664.495\, a_1 + 77898.7\, a_2 - 94741.8\, a_3 &= 7533580 \\
1603.19\, a_0 - 82282.3\, a_1 - 94741.8\, a_2 + 8523670\, a_3 &= -10567800
\end{aligned}\right\}$$

Solving these equations gives

$$a_0 = 3098.7 \qquad a_1 = 374.93$$

$$a_2 = 141.29 \qquad a_3 = 3.367$$

It can be seen that these numbers are correct only to within orders of magnitude. On the other hand, it is not these coefficients which have direct physical significance; rather, it is the damping and frequency of each of the components making up the oscillation which are important. In order to find these numbers, the following equation was set up

$$\lambda^4 + 3.367\, \lambda^3 + 141.29\, \lambda^2 + 374.93\, \lambda + 3098.7 = 0$$

and solved to find the roots:

$$\lambda = \begin{cases} -0.046 \pm 10.7\, i \\ -1.64 \pm 4.96\, i \end{cases}$$

The true roots are obtained by solving the equation

$$\lambda^4 + 2.000\, \lambda^3 + 132.87\, \lambda^2 + 219.32\, \lambda + 2544.9 = 0$$

This gives

$$\lambda = \begin{cases} \pm\ 10.5\ i \\ -\ 1.00 \pm 4.71\ i \end{cases}$$

Thus, the frequencies of the oscillation have been found quite accurately, as has the damping parameter of the undamped component. The only large effect of the errors in the coefficients a_0, a_1, a_2, and a_3 is in the damping of one of the components. The apparent ill-conditioning of the problem with respect to this parameter is not too surprising, for after all, either the true or calculated value of this damping is large enough that the corresponding component of the motion is masked by the undamped component over a good part of the run. This may be seen best, perhaps, from figure 5 in which the solution of equation (32), using both the true and calculated values of the parameters, has been plotted. It can be seen that the two curves do not differ by very much, indicating that the fit could not be much improved.

CONCLUDING REMARKS

A general theory of the so-called "equations-of-motion" methods for the analysis of dynamical systems has been presented. It has been shown that, when looked at from a new point of view, all such methods can be generalized so as to apply to linear and nonlinear systems alike. Using this theory, it has also been shown how new methods can be developed in order to satisfy the requirements of particular problems.

One new method has been described in detail. In certain cases, it reduces to one which is very similar to the well-known Fourier transform method (ref. 3) but in all cases has certain advantages over this latter method and over other methods heretofore used. Its superiority is based on two facts. First, there is the heavy dependence on the initial conditions which occurs when using most of the previously known equations-of-motion methods; this dependence is entirely eliminated in the new method. This superiority manifests itself particularly when systems of higher order than the second are considered. If, for definiteness, a fourth-order system is considered, before the Fourier transform method (for example) can be applied, it is necessary to evaluate the test data and their first three time derivatives at the initial point. Accurate evaluation of all these derivatives is practically impossible, however, with the type of data obtained from most aerodynamic experiments.

The second fact upon which the superiority of the proposed method rests is that most of the equations-of-motion methods used to this time demand an infinitely long record for their rigorous application. For some years now, questions about the errors introduced into an analysis

of a system by the finite length of records available have been asked, but answers have not been offered. The second principal advantage of the method described herein is that such questions are side-stepped completely: There is no error at all from this source, since it is assumed from the start that only a finite record is available. Because of this feature, the method avoids a further limitation of the Fourier transform method (apparently the most accurate of all well-known methods of this type), which cannot be applied at all to some systems (e.g., unstable ones) without time-consuming and sometimes ineffective special devices, since the Fourier integrals of the data simply do not exist.

The single exception to these remarks is the derivative method (refs. 2 and 3). The derivative method does not weight the initial conditions and does not depend on an infinite interval for its application. In addition, the derivative method has in the past been considered as the only well-known method which applies to nonlinear as well as linear systems. (Other methods are described in references 2 and 10, but the derivative method appears to be the only one with such general applicability as we are discussing here.) There are, however, a number of very serious objections to the derivative method. First of all, there is the inordinate amount of time and labor which must be expended in its application, principally because of the necessity for calculating time derivatives of the data. Second, and most important, is the question of accuracy. The accurate calculation of the derivatives needed for the method is most difficult, and this calculation is a large source of error. Besides, even if the derivatives could be computed with the requisite accuracy, the derivative method appears often to lead to badly conditioned equations, as pointed out in reference 2; because of this, many problems have been found for which the derivative method has been shown to lead to extremely large errors. The method proposed herein is subject to none of these weaknesses. The time required for its application is far less than that needed for the derivative method; in addition, it appears to be well suited to machine computation. Naturally, derivatives need not be calculated, and the method shares the properties of the Fourier transform method which cause it to lead to fairly well-conditioned equations.

Ames Aeronautical Laboratory
National Advisory Committee for Aeronautics
Moffett Field, Calif., July 14, 1954

APPENDIX

NUMERICAL EVALUATION OF INTEGRALS OF THE FORM $\int_a^b x(t)y(t)dt$

In 1928, Filon (ref. 8 - see also ref. 9, pp. 67-72) published a generalization of Simpson's rule for evaluation of integrals of the form

$$\int_a^b x(t) \sin \omega t dt$$
$$\int_a^b x(t) \cos \omega t dt$$

where $x(t)$ represents numerical data. Filon's method, in contrast to Simpson's, has the distinction of giving results whose errors are independent of the frequency ω, depending only on how closely $x(t)$ can be fitted to a sequence of parabolas. This method will be generalized to apply to integrals of the form

$$\int_a^b x(t)y(t)dt$$

where $y(t)$ is known exactly (for application to the method described in the body of this report, $y(t)$ is one of the method functions), while $x(t)$ is given tabularly.

Suppose the interval (a,b) is divided into $2h$ equal parts by points $t_o = a < t_1 < . . . < t_{2h} = b$, where $t_{n+1} - t_n = \Delta t$ = constant. Then, a formula of the form

$$\int_a^b x(t)y(t)dt \approx \Delta t \sum_{n=o}^{2h} \Gamma_n(y)x(t_n) \tag{33}$$

will be sought, where the Γ_n are constants which depend only on the function y. These constants will be determined by the condition that the formula (33) will give the integral exactly in the cases where $x(t)$ is a constant, a linear function or a quadratic function of t. Suppose first that the interval (a,b) is divided into two parts only by points t_o, t_1, and t_2; since formula (33) is to be exact if $x(t) = 1$, $(t - t_1)$, or $(t - t_1)^2$, we have

$$\left.\begin{aligned}
\Gamma_0(y) + \Gamma_1(y) + \Gamma_2(y) &= \frac{1}{\Delta t}\int_{t_0}^{t_2} y(t)dt \\
-(\Delta t)\Gamma_0(y) + (\Delta t)\Gamma_2(y) &= \frac{1}{\Delta t}\int_{t_0}^{t_2} (t - t_1)y(t)dt \\
(\Delta t)^2\Gamma_0(y) + (\Delta t)^2\Gamma_2(y) &= \frac{1}{\Delta t}\int_{t_0}^{t_2} (t - t_1)^2 y(t)dt
\end{aligned}\right\} \tag{34}$$

Equations (34) can be solved for Γ_0, Γ_1, Γ_2 to obtain

$$\left.\begin{aligned}
\Gamma_0(y) &= \frac{1}{2(\Delta t)^3}\int_{t_0}^{t_2} (t - t_1)^2 y(t)dt - \frac{1}{2(\Delta t)^2}\int_{t_0}^{t_2} (t - t_1)y(t)dt \\
\Gamma_1(y) &= \frac{1}{\Delta t}\int_{t_0}^{t_2} y(t)dt - \frac{1}{(\Delta t)^3}\int_{t_0}^{t_2} (t - t_1)^2 y(t)dt \\
\Gamma_2(y) &= \frac{1}{2(\Delta t)^3}\int_{t_0}^{t_2} (t - t_1)^2 y(t)dt + \frac{1}{2(\Delta t)^2}\int_{t_0}^{t_2} (t - t_1)y(t)dt
\end{aligned}\right\} \tag{35}$$

Now, if (a,b) is divided into $2h(>2)$ parts, the integral is written as follows:

$$\int_a^b x(t)y(t)dt = \int_{t_0}^{t_2} x(t)y(t)dt + \int_{t_2}^{t_4} x(t)y(t)dt + \ldots +$$

$$\int_{t_{2h-2}}^{t_{2h}} x(t)y(t)dt \tag{36}$$

and equations (35) can be used to evaluate each of the integrals on the right-hand side. Calling

$$J_\rho(y) = \frac{1}{\Delta t}\int_{t_{\rho-1}}^{t_{\rho+1}} y(t)dt$$

$$K_\rho(y) = \frac{1}{2(\Delta t)^2}\int_{t_{\rho-1}}^{t_{\rho+1}} (t - t_\rho)y(t)dt$$

$$L_\rho(y) = \frac{1}{2(\Delta t)^3}\int_{t_{\rho-1}}^{t_{\rho+1}} (t - t_\rho)^2 y(t)dt$$

we obtain from (33), (35) and (36) that

$$\int_a^b x(t)y(t)dt \approx \Delta t \sum_{n=0}^{2h} \Gamma_n(y)x(t_n)$$

where

$$\left.\begin{aligned}
\Gamma_0(y) &= L_1(y) - K_1(y) \\
\Gamma_{2\rho-1}(y) &= J_{2\rho-1}(y) - 2L_{2\rho-1}(y),\ \rho = 1, 2, \ldots, h \\
\Gamma_{2\rho}(y) &= L_{2\rho-1}(y) + K_{2\rho-1}(y) + L_{2\rho+1}(y) - K_{2\rho+1}(y), \\
&\qquad \rho = 1, 2, \ldots, h - 1 \\
\Gamma_{2h}(y) &= L_{2h-1}(y) + K_{2h-1}(y)
\end{aligned}\right\} \tag{37}$$

It should be noted that if $y(t)$ is identically unity,

$$J_\rho(1) = 2$$

$$K_\rho(1) = 0$$

$$L_\rho(1) = 1/3$$

where the relations $t_{\rho-1} = t_\rho - \Delta t$, $t_{\rho+1} = t_\rho + \Delta t$ have been used repeatedly. Hence, from equations (37),

$$\Gamma_0(1) = 1/3$$

$$\Gamma_{2\rho-1}(1) = 4/3, \ \rho = 1, 2, \ldots, h$$

$$\Gamma_{2\rho}(1) = 2/3, \ \rho = 1, 2, \ldots, h - 1$$

$$\Gamma_{2h}(1) = 1/3$$

which exactly describes Simpson's rule.

Equations (37), with $y(t)$ being chosen equal to the method functions, were used to calculate the numbers displayed in tables I, III, and V.

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TABLE I.- CALCULATIONS OF THE INTEGRALS NEEDED FOR EXAMPLE I

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
n	t_n	α_n	α_n^2	α_n^3	$\Gamma_n(y_2)$	$\frac{1}{\omega_2}\Gamma_n(\dot{y}_2)$	$\frac{1}{\omega_2^2}\Gamma_n(\ddot{y}_2)$	$\Gamma_n(y_3)$	$\frac{1}{\omega_3}\Gamma_n(\dot{y}_3)$	$\frac{1}{\omega_3^2}\Gamma_n(\ddot{y}_3)$	$\Gamma_n(y_4)$	$\frac{1}{\omega_4}\Gamma_n(\dot{y}_4)$	$\frac{1}{\omega_4^2}\Gamma_n(\ddot{y}_4)$	$\Gamma_n(y_5)$	$\frac{1}{\omega_5}\Gamma_n(\dot{y}_5)$	$\frac{1}{\omega_5^2}\Gamma_n(\ddot{y}_5)$	$\Gamma_n(y_6)$	$\frac{1}{\omega_6}\Gamma_n(\dot{y}_6)$	$\frac{1}{\omega_6^2}\Gamma_n(\ddot{y}_6)$	$\Gamma_n(y_7)$	$\frac{1}{\omega_7}\Gamma_n(\dot{y}_7)$	$\frac{1}{\omega_7^2}\Gamma_n(\ddot{y}_7)$	$\Gamma_n(y_8)$
0	0	0	0	0	0.0008	0.0002	-0.6699	0.0019	0.0006	0.6743	-0.0032	0.0014	0.6795	-0.0049	0.0026	0.6863	-0.0069	0.0045	0.6944	-0.0094	0.0072	0.7041	-0.0117
1	0.05	0.0455	0.00207	0.000094	.0098	.2081	2.6273	.0231	.3172	2.5741	.0389	.4080	2.5112	.0602	.5024	2.4259	.0857	.5920	2.3237	.1171	.6806	2.1983	.1483
2	.10	.0833	.00694	.000578	.0148	.2070	1.2743	.0348	.3134	1.1942	.0585	.3994	1.0994	.0907	.4853	.9706	.1292	.5618	.8164	.1764	.6303	.6277	.2231
3	.15	.1070	.01145	.001225	.0741	.6038	2.3702	.1706	.8790	1.9844	.2787	1.0681	1.5520	.4155	1.2129	1.0048	.5647	1.2879	.4080	.7263	1.2873	-.2386	.8647
4	.20	.1140	.01300	.001482	.0623	.3938	1.0840	.1418	.5550	.7661	.2284	.6462	.4199	.3333	.6863	0	.4406	.6604	-.4292	.5455	.5619	-.8487	.6220
5	.25	.1053	.01109	.001168	.1964	.9405	1.8810	.4316	1.2394	.9401	.6667	1.3202	0	.9179	1.2129	-1.0048	1.1277	.9220	-1.8441	1.2694	.4670	-2.4110	1.3074
6	.30	.0835	.00697	.000582	.1364	.5420	.7876	.2927	.6694	.1626	.4383	.6462	-.4199	.5760	.4853	-.9706	.6636	.2146	-1.3209	.6794	-.1294	-1.3843	.6220
7	.35	.0530	.00281	.000149	.3648	1.1851	1.2077	.7466	1.3159	-.3196	1.0547	1.0681	-1.5520	1.2731	.5024	-2.4259	1.3106	-.2040	-2.5758	1.1444	-.8710	-1.9108	.8647
8	.40	.0188	.00035	.000007	.2298	.6371	.4140	.4529	.6305	-.4782	.6082	.3994	-1.0994	.6765	0	-1.3727	.6142	-.4082	-1.1236	.4297	-.6773	-.3854	.2231
9	.45	-.0148	.00022	-.000003	.5626	1.3137	.4161	1.0432	1.0909	-1.5060	1.2945	.4080	-2.5112	1.2731	-.5024	-2.4259	.9627	-1.1618	-1.1840	.4898	-1.2434	.7076	.1483
10	.50	-.0413	.00171	-.000071	.3333	.6699	0	.5857	.4472	-1.0095	.6731	0	-1.3589	.5760	-.4853	-.9706	.3333	-.6944	0	.0731	-.4744	1.0407	-.0235
11	.55	-.0620	.00384	-.000238	.7707	1.3137	-.4161	1.2535	.6160	-2.3475	1.2945	-.4080	-2.5112	.9179	-1.2129	-1.0048	.3707	-1.1618	1.1840	.0313	-.2376	2.5416	.1483
12	.60	-.0720	.00518	-.000373	.4368	.6371	-.4140	.6607	.1614	-1.3095	.6082	-.3994	-1.0994	.3333	-.6863	0	.0524	-.4082	1.1236	.0050	.2544	1.3132	.2231
13	.65	-.0718	.00516	-.000370	.9686	1.1851	-1.2077	1.3294	0	-2.6511	1.0547	-1.0681	-1.5520	.4155	-1.2129	1.0048	.0227	-.2040	2.5758	.2771	1.0317	1.5582	.8647
14	.70	-.0612	.00375	-.000230	.5302	.5420	-.7876	.6607	-.1614	-1.3095	.4383	-.6462	-.4199	.0907	-.4853	.9706	.0031	.2146	1.3209	.3008	.7011	.1299	.6220
15	.75	-.0448	.00201	-.000090	1.1369	.9405	-1.8810	1.2535	-.6160	-2.3475	.6667	-1.3202	0	.0602	-.5024	2.4259	.2056	.9220	1.8441	.9548	1.1573	-1.1525	1.3074
16	.80	-.0233	.00054	-.000013	.6043	.3938	-1.0840	.5857	-.4472	-1.0095	.2284	-.6462	.4199	-.0098	0	1.3727	.2260	.6604	.4292	.6327	.3707	-1.1974	.6220
17	.85	-.0015	0	0	1.2592	.6038	-2.3702	1.0432	-1.0909	-1.5060	.2787	-1.0681	1.5520	.0602	.5024	2.4259	.7687	1.2879	-.4080	1.3131	0	-2.5856	.8647
18	.90	.0183	.00033	.000006	.6519	.2070	-1.2743	.4529	-.6305	-.4782	.0585	-.3994	1.0994	.0907	.4853	.9706	.5374	.5618	-.8164	.6327	-.3707	-1.1974	.2231
19	.95	.0335	.00112	.000038	1.3235	.2081	-2.6273	.7466	-1.3159	-.3196	.0389	-.4080	2.5112	.4155	1.2129	1.0048	1.2476	.5920	-2.3237	.9548	-1.1573	-1.1525	.1483
20	1.00	.0445	.00198	.000088	.6683	0	-1.3399	.2927	-.6694	.1626	-.0064	0	1.3589	.3333	.6863	0	.6806	0	-1.3889	.3008	-.7011	.1299	-.0235
21	1.05	.0480	.00230	.000110	1.3235	-.2081	-2.6273	.4316	-1.2394	.9401	.0389	.4080	2.5112	.9179	1.2129	-1.0048	1.2476	-.5920	-2.3237	.2771	-1.0317	1.5582	.1483
22	1.10	.0460	.00212	.000098	.6519	-.2070	-1.2743	.1418	-.5550	.7661	.0585	.3994	1.0994	.5760	.4853	-.9706	.5374	-.5618	-.8164	.0050	-.2544	1.3132	.2231
23	1.15	.0377	.00142	.000054	1.2592	-.6038	-2.3702	.1706	-.8790	1.9844	.2787	1.0681	1.5520	1.2731	.5024	-2.4259	.7687	-1.2879	-.4080	-.0313	.2376	2.5416	.8647
24	1.20	.0265	.00070	.000019	.6043	-.3938	-1.0840	.0348	-.3134	1.1942	.2284	.6462	.4199	.6765	0	-1.3727	.2260	-.6604	.4292	.0731	.4744	1.0407	.6220
25	1.25	.0125	.00016	.000002	1.1369	-.9405	-1.8810	.0231	-.3172	2.5741	.6667	1.3202	0	1.2731	-.5024	-2.4259	.2056	-.9220	1.8441	.4898	1.2434	.7076	1.3074
26	1.30	-.0010	0	0	.5302	-.5420	-.7876	.0019	-.0006	.6743	.4383	.6462	-.4199	.5760	-.4853	-.9706	.0031	-.2146	1.3209	.4297	.6773	-.3854	.6220
27	1.35	-.0132	.00017	-.000002	.9686	-1.1851	-1.2077	0	0	0	1.0547	1.0681	-1.5520	.9179	-1.2129	-1.0048	.0227	.2040	2.5758	1.1444	.8710	-1.9108	.8647
28	1.40	-.0233	.00054	-.000013	.4368	-.6371	-.4140	0	0	0	.6082	.3994	-1.0994	.3333	-.6863	0	.0524	.4082	1.1236	.6794	.1294	-1.3843	.2231
29	1.45	-.0290	.00084	-.000024	.7707	-1.3137	-.4161	0	0	0	1.2945	.4080	-2.5112	.4155	-1.2129	1.0048	.3707	1.1618	1.1840	1.2694	-.4670	-2.4110	.1483
30	1.50	-.0303	.00092	-.000028	.3333	-.6699	0	0	0	0	.6731	0	-1.3589	.0907	-.4853	.9706	.3333	.6944	0	.5455	-.5619	-.8487	-.0235
31	1.55	-.0285	.00081	-.000023	.5626	-1.3137	.4161	0	0	0	1.2945	-.4080	-2.5112	.0602	-.5024	2.4259	.9627	1.1618	-1.1840	.7263	-1.2873	-.2386	.1483
32	1.60	-.0232	.00054	-.000013	.2298	-.6371	.4140	0	0	0	.6082	-.3994	-1.0994	-.0049	-.0026	.6863	.6142	.4082	-1.1236	.1764	-.6303	.6277	.2231
33	1.65	-.0152	.00023	-.000003	.3648	-1.1851	1.2077	0	0	0	1.0547	-1.0681	-1.5520	0	0	0	1.3106	.2040	-2.5758	.1171	-.6806	2.1983	.8647
34	1.70	-.0065	.00004	0	.1364	-.5420	.7876	0	0	0	.4383	-.6462	-.4199	0	0	0	.6636	-.2146	-1.3209	-.0094	-.0072	.7041	.6220
35	1.75	.0025	.00001	0	.1964	-.9405	1.8810	0	0	0	.6667	-1.3202	0	0	0	0	1.1277	-.9220	-1.8441	0	0	0	1.3074
36	1.80	.0100	.00010	.000001	.0623	-.3938	1.0840	0	0	0	.2284	-.6462	.4199	0	0	0	.4406	-.6604	-.4292	0	0	0	.6220
37	1.85	.0158	.00025	.000004	.0741	-.6038	2.3702	0	0	0	.2787	-1.0681	1.5520	0	0	0	.5647	-1.2879	.4080	0	0	0	.8647
38	1.90	.0188	.00035	.000007	.0148	-.2070	1.2743	0	0	0	.0585	-.3994	1.0994	0	0	0	.1292	-.5618	.8164	0	0	0	.2231
39	1.95	.0195	.00038	.000007	.0098	-.2081	2.6273	0	0	0	.0389	-.4080	2.5112	0	0	0	.0857	-.5920	2.3237	0	0	0	.1483
40	2.00	.0180	.00032	.000006	.0008	-.0002	.6699	0	0	0	-.0032	-.0014	.6795	0	0	0	-.0069	-.0045	.6944	0	0	0	-.0117
				Σ ③ × ○ =	-.0215	.0961	.8724	-.1346	.2151	1.5143	-.1872	.7911	1.5349	.3022	1.0268	-.5090	.4442	.2094	-.9905	.2778	.0979	-.4695	.3006
				Σ ④ × ○ =	.03343			.03794			.04329			.04476			.04801			.04279			.05680
				Σ ⑤ × ○ =	-.000053			.000005			.000705			.002363			.003463			.003606			.003532

○ Denotes the column in which the numbers appear

TABLE I.- CALCULATIONS OF THE INTEGRALS NEEDED FOR EXAMPLE I - Continued

	25	26	27	28	29	30	31	32	33	34
n	$\frac{1}{\omega_8}\Gamma_n(\dot{y}_8)$	$\frac{1}{\omega_8^2}\Gamma_n(\ddot{y}_8)$	$\Gamma_n(y_9)$	$\frac{1}{\omega_9}\Gamma_n(\dot{y}_9)$	$\frac{1}{\omega_9^2}\Gamma_n(\ddot{y}_9)$	$\Gamma_n(y_{10})$	$\frac{1}{\omega_{10}}\Gamma_n(\dot{y}_{10})$	$\frac{1}{\omega_{10}^2}\Gamma_n(\ddot{y}_{10})$	$\Gamma_n(y_{11})$	$\frac{1}{\omega_{11}}\Gamma_n(\dot{y}_{11})$
0	0.0104	0.7136	-0.0141	0.0141	0.7230	-0.0171	0.0197	0.7352	-0.0202	0.0265
1	.7532	2.0734	.1804	.8160	1.9449	.2237	.8859	1.7718	.2699	.9457
2	.6787	.4410	.2706	.7120	.2511	.3333	.7352	0	.3982	.7363
3	1.2187	-.7920	.9840	1.0994	-1.2695	1.1096	.8859	-1.7718	1.2012	.6173
4	.4194	-1.1546	.6730	.2473	-1.3588	.7009	0	-1.4704	.6846	-.2557
5	0	-2.5629	1.2631	-.4342	-2.3858	1.1096	-.8859	-1.7718	.8778	-1.1601
6	-.4194	-1.1546	.5141	-.6261	-.7230	.3333	-.7352	0	.1464	-.6474
7	-1.2187	-.7920	.5564	-1.2502	.4409	.2237	-.8859	1.7718	.0588	-.2144
8	-.6787	.4410	.0564	-.4647	1.1077	-.0343	0	1.4704	.0470	.4806
9	-.7532	2.0734	.0319	0	2.5389	.2237	.8859	1.7718	.6667	1.2345
10	0	1.4272	.0564	.4647	1.1077	.3333	.7352	0	.6197	.4806
11	.7532	2.0734	.5564	1.2502	.4409	1.1096	.8859	-1.7718	1.2745	-.2144
12	.6787	.4410	.5141	.6261	-.7230	.7009	0	-1.4704	.5202	-.6474
13	1.2187	-.7920	1.2631	.4342	-2.3858	1.1096	-.8859	-1.7718	.4556	-1.1601
14	.4194	-1.1546	.6730	-.2473	-1.3588	.3333	-.7352	0	-.0179	-.2557
15	0	-2.5629	.9840	-1.0994	-1.2695	.2237	-.8859	1.7718	.1321	.6173
16	-.4194	-1.1546	.2706	-.7120	.2511	-.0343	0	1.4704	.2684	.7363
17	-1.2187	-.7920	.1804	-.8160	1.9449	.2237	.8859	1.7718	1.0634	.9457
18	-.6787	.4410	-.0282	0	1.4460	.3333	.7352	0	.7071	0
19	-.7532	2.0734	.1804	.8160	1.9449	1.1096	.8859	-1.7718	1.0634	-.9457
20	0	1.4272	.2706	.7120	.2511	.7009	0	-1.4704	.2684	-.7363
21	.7532	2.0734	.9840	1.0994	-1.2695	1.1096	-.8859	-1.7718	.1321	-.6173
22	.6787	.4410	.6730	.2473	-1.3588	.3333	-.7352	0	-.0179	.2557
23	1.2187	-.7920	1.2631	-.4342	-2.3858	.2237	-.8859	1.7718	.4556	1.1601
24	.4194	-1.1546	.5141	-.6261	-.7230	-.0343	0	1.4704	.5202	.6474
25	0	-2.5629	.5564	-1.2502	.4409	.2237	.8859	1.7718	1.2745	.2144
26	-.4194	-1.1546	.0564	-.4647	1.1077	.3333	.7352	0	.6197	-.4806
27	-1.2187	-.7920	.0319	0	2.5389	1.1096	.8859	-1.7718	.6667	-1.2345
28	-.6787	.4410	.0564	.4647	1.1077	.7009	0	-1.4704	.0470	-.4806
29	-.7532	2.0734	.5564	1.2502	.4409	1.1096	-.8859	-1.7718	.0588	.2144
30	0	1.4272	.5141	.6261	-.7230	.3333	-.7352	0	.1464	.6474
31	.7532	2.0734	1.2631	.4342	-2.3858	.2237	-.8859	1.7718	.8778	1.1601
32	.6787	.4410	.6730	-.2473	-1.3588	-.0343	0	1.4704	.6846	.2557
33	1.2187	-.7920	.9840	-1.0994	-1.2695	.2237	.8859	1.7718	1.2012	-.6173
34	.4194	-1.1546	.2706	-.7120	.2511	.3333	.7352	0	.3982	-.7363
35	0	-2.5629	.1804	-.8160	1.9449	1.1096	.8859	-1.7718	.2699	-.9457
36	-.4194	-1.1546	-.0141	-.0141	.7230	.7009	0	-1.4704	-.0202	-.0265
37	-1.2187	-.7920	0	0	0	1.1096	-.8859	-1.7718	0	0
38	-.6787	.4410	0	0	0	.3333	-.7352	0	0	0
39	-.7532	2.0734	0	0	0	.2237	-.8859	1.7718	0	0
40	-.0104	.7136	0	0	0	-.0171	-.0197	.7352	0	0
Σ ③ × ○ =	.0021	-.4171	.2248	0	-.2498	.2528	-.0130	-.2250	.1984	-.0068
Σ ④ × ○ =			.06677			.06467			.05354	
Σ ⑤ × ○ =			.003406			.003307			.003216	

	35	36	37	38	39	40	41	42	43
n	$\frac{1}{\omega_{11}^2}\Gamma_n(\ddot{y}_{11})$	$\Gamma_n(y_{12})$	$\frac{1}{\omega_{12}}\Gamma_n(\dot{y}_{12})$	$\frac{1}{\omega_{12}^2}\Gamma_n(\ddot{y}_{12})$	$\Gamma_n(y_{13})$	$\frac{1}{\omega_{13}}\Gamma_n(\dot{y}_{13})$	$\frac{1}{\omega_{13}^2}\Gamma_n(\ddot{y}_{13})$	$\Gamma_n(y_{14})$	$\frac{1}{\omega_{14}}\Gamma_n(\dot{y}_{14})$
0	0.7476	-0.0227	0.0328	0.7574	-0.0262	0.0436	0.7715	-0.0278	0.0497
1	1.5871	.3085	.9859	1.4325	.3685	1.0329	1.1927	.3991	1.0504
2	-.2596	.4504	.7203	-.4681	.5262	.6681	-.7715	.5620	.6295
3	-2.1382	1.2461	.3766	-2.3179	1.2630	0	-2.3855	1.2489	-.1844
4	-1.4050	.6397	-.4452	-1.2255	.5262	-.6681	-.7715	.4535	-.7400
5	-.8445	.6667	-1.2186	0	.3685	-1.0329	1.1927	.2498	-.8336
6	.7476	.0270	-.4452	1.2255	-.0524	0	1.5429	-.0367	.2404
7	2.4315	.0872	.3766	2.3179	.3685	1.0329	1.1927	.5745	1.1644
8	1.1454	.2163	.7203	.4681	.5262	.6681	-.7715	.6481	.4573
9	0	1.0248	.9859	-1.4325	1.2630	0	-2.3855	1.1919	-.5352
10	-1.1454	.7120	0	-1.5148	.5262	-.6681	-.7715	.3333	-.7780
11	-2.4315	1.0248	-.9859	-1.4325	.3685	-1.0329	1.1927	.1414	-.5352
12	-.7476	.2163	-.7203	.4681	-.0524	0	1.5429	.0186	.4573
13	.8445	.0872	-.3766	2.3179	.3685	1.0329	1.1927	.7589	1.1644
14	1.4050	.0270	.4452	1.2255	.5262	.6681	-.7715	.7033	.2404
15	2.1382	.6667	1.2186	0	1.2630	0	-2.3855	1.0835	-.8336
16	.2596	.6397	.4452	-1.2255	.5262	-.6681	-.7715	.2131	-.7400
17	-1.5871	1.2461	-.3766	-2.3179	.3685	-1.0329	1.1927	.0845	-.1844
18	-1.4952	.4504	-.7203	-.4681	-.0524	0	1.5429	.1047	.6295
19	-1.5871	.3085	-.9859	1.4325	.3685	1.0329	1.1927	.9343	1.0504
20	.2596	-.0454	0	1.5148	.5262	.6681	-.7715	.7224	0
21	2.1382	.3085	.9859	1.4325	1.2630	0	-2.3855	.9343	-1.0504
22	1.4050	.4504	.7203	-.4681	.5262	-.6681	-.7715	.1047	-.6295
23	.8445	1.2461	.3766	-2.3179	.3685	-1.0329	1.1927	.0845	.1844
24	-.7476	.6397	-.4452	-1.2255	-.0524	0	1.5429	.2131	.7400
25	-2.4315	.6667	-1.2186	0	.3685	1.0329	1.1927	1.0835	.8336
26	-1.1454	.0270	-.4452	1.2255	.5262	.6681	-.7715	.7033	-.2404
27	0	.0872	.3766	2.3179	1.2630	0	-2.3855	.7589	-1.1644
28	1.1454	.2163	.7203	.4681	.5262	-.6681	-.7715	.0186	-.4573
29	2.4315	1.0248	.9859	-1.4325	.3685	-1.0329	1.1927	.1414	.5352
30	.7476	.7120	0	-1.5148	-.0524	0	1.5429	.3333	.7780
31	-.8445	1.0248	-.9859	-1.4325	.3685	1.0329	1.1927	1.1919	.5352
32	-1.4050	.2163	-.7203	.4681	.5262	.6681	-.7715	.6481	-.4573
33	-2.1382	.0872	-.3766	2.3179	1.2630	0	-2.3855	.5745	-1.1644
34	-.2596	.0270	.4452	1.2255	.5262	-.6681	-.7715	-.0367	-.2404
35	1.5871	.6667	1.2186	0	.3685	-1.0329	1.1927	.2498	.8336
36	.7476	.6397	.4452	-1.2255	-.0262	-.0436	.7715	.4535	.7400
37	0	1.2461	-.3766	-2.3179	0	0	0	1.2489	.1844
38	0	.4504	-.7203	-.4681	0	0	0	.5620	-.6295
39	0	.3085	-.9859	1.4325	0	0	0	.3991	-1.0504
40	0	-.0227	-.0328	.7574	0	0	0	-.0278	-.0497
Σ ③ × ○ =	-.1452	.2337	-.0165	-.1485	.1855	-.0125	-.0939	.2208	-.0191
Σ ④ × ○ =		.05064			.04734			.04763	
Σ ⑤ × ○ =		.003194			.002871			.002498	

NACA

TABLE I.- CALCULATIONS OF THE INTEGRALS NEEDED FOR EXAMPLE I - Concluded

	44	45	46	47	48	49	50
n	$\frac{1}{\omega_{14}^2}\Gamma_n(\ddot{y}_{14})$	$\Gamma_n(y_{15})$	$\frac{1}{\omega_{15}}\Gamma_n(\dot{y}_{15})$	$\frac{1}{\omega_{15}^2}\Gamma_n(\ddot{y}_{15})$	$\Gamma_n(y_{16})$	$\frac{1}{\omega_{16}}\Gamma_n(\dot{y}_{16})$	$\frac{1}{\omega_{16}^2}\Gamma_n(\ddot{y}_{16})$
0	0.7780	-0.0295	0.0568	0.7848	-0.0321	0.0703	0.7951
1	1.0705	.4331	1.0651	.9344	.4914	1.0788	.7010
2	-.9146	.5991	.5774	-1.0631	.6549	.4673	-1.2864
3	-2.3288	1.2167	-.3776	-2.2001	1.1255	-.6667	-1.8354
4	-.4809	.3657	-.7822	-.1296	.2105	-.7561	.4914
5	1.6673	.1552	-.5536	2.0457	.0995	0	2.2686
6	1.4799	.0237	.4821	1.2387	.2105	.7561	.4914
7	.3689	.8094	1.1275	-.5710	1.1255	.6667	-1.8354
8	-1.2589	.7204	.1292	-1.5483	.6549	-.4673	-1.2864
9	-2.1009	.9847	-.9737	-1.2723	.4914	-1.0788	.7010
10	0	.1187	-.6570	.8585	-.0642	0	1.5901
11	2.1009	.0931	.1914	2.2944	.4914	1.0788	.7010
12	1.2589	.2370	.7608	.3853	.6549	.4673	-1.2864
13	-.3689	1.1256	.7144	-1.8356	1.1255	-.6667	-1.8354
14	-1.4799	.6785	-.3735	-1.3805	.2105	-.7561	.4914
15	-1.6673	.6186	-1.1591	.1921	.0995	0	2.2686
16	.4809	-.0378	-.2548	1.4846	.2105	.7561	.4914
17	2.3288	.2728	.8557	1.5754	1.1255	.6667	-1.8354
18	.9147	.4910	.7187	-.6305	.6549	-.4673	-1.2864
19	-1.0705	1.2482	0	-2.3261	.4914	-1.0788	.7010
20	-1.5561	.4910	-.7187	-.6305	-.0642	0	1.5901
21	-1.0705	.2728	-.8557	1.5754	.4914	1.0788	.7010
22	.9147	-.0378	.2548	1.4846	.6549	.4673	-1.2864
23	2.3288	.6186	1.1591	.1921	1.1255	-.6667	-1.8354
24	.4809	.6785	.3735	-1.3805	.2105	-.7561	.4914
25	-1.6673	1.1256	-.7144	-1.8356	.0995	0	2.2686
26	-1.4799	.2370	-.7608	.3853	.2105	.7561	.4914
27	-.3689	.0931	-.1914	2.2944	1.1255	.6667	-1.8354
28	1.2589	.1187	.6570	.8585	.6549	-.4673	-1.2864
29	2.1009	.9847	.9737	-1.2723	.4914	-1.0788	.7010
30	0	.7204	-.1292	-1.5483	-.0642	0	1.5901
31	-2.1009	.8094	-1.1275	-.5710	.4914	1.0788	.7010
32	-1.2589	.0237	-.4821	1.2387	.6549	.4673	-1.2864
33	.3689	.1552	.5536	2.0457	1.1255	-.6667	-1.8354
34	1.4799	.3657	.7822	-.1296	.2105	-.7561	.4914
35	1.6673	1.2167	.3776	-2.2001	.0995	0	2.2686
36	-.4809	.5991	-.5774	-1.0631	.2105	.7561	.4914
37	-2.3288	.4331	-1.0651	.9344	1.1255	.6667	-1.8354
38	-.9146	-.0295	-.0568	.7848	.6549	-.4673	-1.2864
39	1.0705	0	0	0	.4914	-1.0788	.7010
40	.7780	0	0	0	-.0321	-.0703	.7951
Σ ③ × ◯ =	-.0970	.1961	-.0186	-.0747	.2141	-.0124	-.0702
Σ ④ × ◯ =		.04680			.04688		
Σ ⑤ × ◯ =		.002096			.001861		

NACA

TABLE II.- SUMS NEEDED FOR THE EVALUATION OF THE PARAMETERS OF EXAMPLE I BY LEAST SQUARES

	1	2	3	4	5	6	7	8	9
ν	ω_ν	ω_ν^2	$\frac{1}{\omega_\nu}\Delta t\int_0^2 \alpha\dot{y}_\nu dt$	① × ③	$\frac{1}{\Delta t}\int_0^2 \alpha y_\nu dt$	$\frac{1}{\Delta t}\int_0^2 \alpha^2 y_\nu dt$	$\frac{1}{\Delta t}\int_0^2 \alpha^3 y_\nu dt$	$\frac{1}{\omega_\nu^2}\Delta t\int_0^2 \alpha\ddot{y}_\nu dt$	② × ⑧
2	1.5708	2.4674	0.0961	0.1510	-0.0215	0.03343	-0.000053	0.8724	2.1526
3	2.4166	5.8400	.2151	.5198	-.1346	.03794	.000005	1.5143	8.8435
4	3.1416	9.8697	.7911	2.4853	-.1872	.04329	.000705	1.5349	15.1490
5	3.9270	15.4213	1.0268	4.0322	.3022	.04476	.002363	-.5090	-7.8494
6	4.7124	22.2067	.2094	.9868	.4442	.04801	.003463	-.9905	-21.9957
7	5.5440	30.7359	.0979	.5428	.2772	.04979	.003606	-.4695	-14.4305
8	6.2832	39.4786	.0021	.0132	.3006	.05680	.003532	-.4171	-16.4665
9	6.9813	48.7385	0	0	.2248	.06677	.003406	-.2498	-12.1749
10	7.8540	61.6853	-.0130	-.1021	.2528	.06467	.003307	-.2250	-13.8792
11	8.7267	76.1553	-.0068	-.0593	.1984	.05354	.003216	-.1452	-11.0577
12	9.4248	88.8269	-.0165	-.1555	.2337	.05064	.003194	-.1485	-13.1908
13	10.4720	109.6628	-.0125	-.1309	.1855	.04734	.002871	-.0939	-10.2973
14	10.9956	120.9032	-.0191	-.2100	.2208	.04763	.002498	-.0970	-11.7276
15	11.5743	133.9644	-.0186	-.2153	.1961	.04680	.002096	-.0747	-10.0071
16	12.5664	157.9144	-.0124	-.1558	.2141	.04688	.001861	-.0702	-11.0856

Σ ④^2 = 24.1669
Σ ④ × ⑤ = 1.05270
Σ ④ × ⑥ = .336760
Σ ④ × ⑦ = .0140294
Σ ④ × ⑨ = -7.01790
Σ ⑤^2 = .885340
Σ ⑤ × ⑥ = .144127

Σ ⑤ × ⑦ = .00906192
Σ ⑤ × ⑨ = -45.5226
Σ ⑥^2 = .0374138
Σ ⑥ × ⑦ = .00189656
Σ ⑥ × ⑨ = -7.00244
Σ ⑦^2 = .000108975
Σ ⑦ × ⑨ = -.459829

TABLE III.- CALCULATION OF THE INTEGRALS NEEDED FOR EXAMPLE II

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
n	τ_n	q_n	a_n	a_n^2	a_n^3	$\Gamma_n(y_2)$	$\frac{1}{\omega_2}\Gamma_n\left(\frac{dy_2}{d\tau}\right)$	$\Gamma_n(y_3)$	$\frac{1}{\omega_3}\Gamma_n\left(\frac{dy_3}{d\tau}\right)$	$\Gamma_n(y_4)$	$\frac{1}{\omega_4}\Gamma_n\left(\frac{dy_4}{d\tau}\right)$	$\Gamma_n(y_5)$	$\frac{1}{\omega_5}\Gamma_n\left(\frac{dy_5}{d\tau}\right)$	$\Gamma_n(y_6)$	$\frac{1}{\omega_6}\Gamma_n\left(\frac{dy_6}{d\tau}\right)$	$\Gamma_n(y_7)$	$\frac{1}{\omega_7}\Gamma_n\left(\frac{dy_7}{d\tau}\right)$	$\Gamma_n(y_8)$	$\frac{1}{\omega_8}\Gamma_n\left(\frac{dy_8}{d\tau}\right)$
0	0	-1.00	0.1000	0.010000	0.0010000	0	0.3337	0.0001	0.3343	0.0002	0.3350	0.0003	0.3359	0.0006	0.3370	0.0009	0.3383	0.0014	0.3397
1	.05	-2.24	.0500	.002500	.0001250	.1045	1.3284	.1605	1.3217	.2081	1.3137	.2591	1.3027	.3095	1.2893	.3621	1.2726	.4080	1.2556
2	.10	-1.74	.0010	.000001	0	.1044	.6593	.1600	.6492	.2070	.6371	.2571	.6206	.3060	.6005	.3562	.5753	.3994	.5497
3	.15	-1.62	-.0390	.001521	-.0000593	.3111	1.2957	.4721	1.2449	.6038	1.1851	.7379	1.1044	.8611	1.0083	.9778	.8914	1.0681	.7760
4	.20	-1.03	-.0740	.005476	-.0004052	.2063	.6348	.3107	.5920	.3938	.5420	.4750	.4750	.5452	.3961	.6057	.3016	.6462	.2100
5	.25	.61	-.0790	.006241	-.0004930	.5099	1.2311	.7563	1.0957	.9405	.9405	1.1044	.7379	1.2250	.5074	1.3006	.2431	1.3202	0
6	.30	1.14	-.0535	.002862	-.0001531	.3030	.5947	.4434	.5005	.5420	.3938	.6206	.2571	.6657	.1054	.6738	-.0624	.6462	-.2100
7	.35	.80	-.0285	.000812	-.0000231	.6962	1.1362	.9966	.8829	1.1851	.6038	1.3027	.2591	1.3219	-.1040	1.2338	-.4780	1.0681	-.7760
8	.40	.49	-.0120	.000144	-.0000017	.3923	.5400	.5503	.3798	.6371	.2070	.6718	0	.6410	-.2083	.5400	-.4078	.3994	-.5497
9	.45	.33	-.0020	.000004	0	.8654	1.0132	1.1789	.6187	1.3137	.2081	1.3027	-.2591	1.1306	-.6928	.7974	-1.0559	.4080	-1.2556
10	.50	.30	.0060	.000036	.0000002	.4720	.4720	.6252	.2371	.6699	0	.6206	-.2571	.4766	-.4766	.2444	-.6310	0	-.6795
11	.55	.35	.0135	.000182	.0000025	1.0132	.8654	1.2927	.3186	1.3137	-.2081	1.1044	-.7379	.6928	-1.1306	.1221	-1.3175	-.4080	-1.2556
12	.60	.47	.0230	.000529	.0000122	.5400	.3923	.6637	.0806	.6371	-.2070	.4750	-.4750	.2083	-.6410	-.1243	-.6652	-.3994	-.5497
13	.65	.60	.0365	.001332	.0000486	1.1362	.6962	1.3314	0	1.1851	-.6038	.7379	-1.1044	.1040	-1.3219	-.5898	-1.1844	-1.0681	-.7760
14	.70	.62	.0515	.002652	.0001366	.5947	.3030	.6637	-.0806	.5420	-.3938	.2571	-.6206	-.1054	-.6657	-.4559	-.5001	-.6462	-.2100
15	.75	.32	.0625	.003906	.0002441	1.2311	.5099	1.2927	-.3186	.9405	-.9405	.2591	-1.3027	-.5074	-1.2250	-1.1249	-.6965	-1.3202	0
16	.80	-.17	.0630	.003969	.0002500	.6348	.2063	.6252	-.2371	.3938	-.5420	0	-.6718	-.3961	-.5452	-.6509	-.1852	-.6462	.2100
17	.85	-.40	.0535	.002862	.0001531	1.2957	.3111	1.1789	-.6187	.6038	-1.1851	-.2591	-1.3027	-1.0083	-.8611	-1.3231	0	-1.0681	.7760
18	.90	-.30	.0440	.001936	.0000852	.6593	.1044	.5503	-.3798	.2070	-.6371	-.2571	-.6206	-.6005	-.3060	-.6509	.1852	-.3994	.5497
19	.95	-.09	.0385	.001482	.0000571	1.3284	.1045	.9966	-.8829	.2081	-1.3137	-.7379	-1.1044	-1.2893	-.3095	-1.1249	.6965	-.4080	1.2556
20	1.00	.11	.0380	.001444	.0000549	.6675	0	.4434	-.5005	0	-.6699	-.4750	-.4750	-.6740	0	-.4559	.5001	0	.6795
21	1.05	.26	.0425	.001806	.0000768	1.3284	-.1045	.7563	-1.0957	-.2081	-1.3137	-1.1044	-.7379	-1.2893	.3095	-.5898	1.1844	.4080	1.2556
22	1.10	.32	.0490	.002401	.0001176	.6593	-.1044	.3107	-.5920	-.2070	-.6371	-.6206	-.2571	-.6005	.3060	-.1243	.6652	.3994	.5497
23	1.15	.23	.0550	.003025	.0001664	1.2957	-.3111	.4721	-1.2449	-.6038	-1.1851	-1.3027	-.2591	-1.0083	.8611	.1221	1.3175	1.0681	.7760
24	1.20	.03	.0570	.003249	.0001852	.6348	-.2063	.1600	-.6492	-.3938	-.5420	-.6718	0	-.3961	.5452	.2444	.6310	.6462	.2100
25	1.25	-.13	.0550	.003025	.0001664	1.2311	-.5099	.1605	-1.3217	-.9405	-.9405	-1.3027	.2591	-.5074	1.2250	.7974	1.0559	1.3202	0
26	1.30	-.15	.0500	.002500	.0001250	.5947	-.3030	.0001	-.3343	-.5420	-.3938	-.6206	.2571	-.1054	.6657	.5400	.4078	.6462	-.2100
27	1.35	-.07	.0465	.002162	.0001005	1.1362	-.6962	0	0	-1.1851	-.6038	-1.1044	.7379	.1040	1.3219	1.2338	.4780	1.0681	-.7760
28	1.40	.04	.0450	.002025	.0000911	.5400	-.3923	0	0	-.6371	-.2070	-.4750	.4750	.2083	.6410	.6738	.0624	.3994	-.5497
29	1.45	.14	.0465	.002162	.0001005	1.0132	-.8654	0	0	-1.3137	-.2081	-.7379	1.1044	.6928	1.1306	1.3006	-.2431	.4080	-1.2556
30	1.50	.17	.0495	.002450	.0001213	.4720	-.4720	0	0	-.6699	0	-.2571	.6206	.4766	.4766	.6057	-.3016	0	-.6795
31	1.55	.13	.0525	.002756	.0001447	.8654	-1.0132	0	0	-1.3137	.2081	-.2591	1.3027	1.1306	.6928	.9778	-.8914	-.4080	-1.2556
32	1.60	.04	.0535	.002862	.0001531	.3923	-.5400	0	0	-.6371	.2070	-.0003	.3359	.6410	.2083	.3562	-.5753	-.3994	-.5497
33	1.65	-.03	.0530	.002809	.0001489	.6962	-1.1362	0	0	-1.1851	.6038	0	0	1.3219	.1040	.3621	-1.2726	-1.0681	-.7760
34	1.70	-.05	.0510	.002601	.0001327	.3030	-.5947	0	0	-.5420	.3938	0	0	.6657	-.1054	.0009	-.3383	-.6462	-.2100
35	1.75	-.02	.0490	.002401	.0001176	.5099	-1.2311	0	0	-.9405	.9405	0	0	1.2250	-.5074	0	0	-1.3202	0
36	1.80	.03	.0480	.002304	.0001106	.2063	-.6348	0	0	-.3938	.5420	0	0	.5452	-.3961	0	0	-.6462	.2100
37	1.85	.08	.0485	.002352	.0001141	.3111	-1.2957	0	0	-.6038	1.1851	0	0	.8611	-1.0083	0	0	-1.0681	.7760
38	1.90	.10	.0500	.002500	.0001250	.1044	-.6593	0	0	-.2070	.6371	0	0	.3060	-.6005	0	0	-.3994	.5497
39	1.95	.09	.0515	.002652	.0001366	.1045	-1.3284	0	0	-.2081	1.3137	0	0	.3095	-1.2893	0	0	-.4080	1.2556
40	2.00	.04	.0525	.002756	.0001447	0	-.3337	0	0	-.0002	.3350	0	0	.0006	-.3370	0	0	-.0014	.3397
Σ ③ × ○ =						2.8046	-3.6366	2.8806	-4.7790	2.0718	-5.8272	1.4802	-6.8972	.7873	-8.0420	-.6023	-8.5924	-2.3114	-8.6146
Σ ④ × ○ =						.90858	-.67276	.31917	-.54119	-.58916	-.39312	-.62799	-.24078	-.20354	-.09202	-.23443	.04498	-.54096	.15638
Σ ⑤ × ○ =						.056655		.032479		-.008983		-.007054		.017034		.017091		.002066	
Σ ⑥ × ○ =						.0021576		.0005784		-.001793		-.0019516		-.0009356		-.0010623		-.0018572	

○ Denotes the column in which the numbers appear

TABLE III.- CALCULATION OF THE INTEGRALS NEEDED FOR EXAMPLE II - Concluded

	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36
n	$\Gamma_n(y_9)$	$\frac{1}{\omega_9}\Gamma_n\left(\frac{dy_9}{d\tau}\right)$	$\Gamma_n(y_{10})$	$\frac{1}{\omega_{10}}\Gamma_n\left(\frac{dy_{10}}{d\tau}\right)$	$\Gamma_n(y_{11})$	$\frac{1}{\omega_{11}}\Gamma_n\left(\frac{dy_{11}}{d\tau}\right)$	$\Gamma_n(y_{12})$	$\frac{1}{\omega_{12}}\Gamma_n\left(\frac{dy_{12}}{d\tau}\right)$	$\Gamma_n(y_{13})$	$\frac{1}{\omega_{13}}\Gamma_n\left(\frac{dy_{13}}{d\tau}\right)$	$\Gamma_n(y_{14})$	$\frac{1}{\omega_{14}}\Gamma_n\left(\frac{dy_{14}}{d\tau}\right)$	$\Gamma_n(y_{15})$	$\frac{1}{\omega_{15}}\Gamma_n\left(\frac{dy_{15}}{d\tau}\right)$	$\Gamma_n(y_{16})$	$\frac{1}{\omega_{16}}\Gamma_n\left(\frac{dy_{16}}{d\tau}\right)$
0	0.0019	0.3412	0.0026	0.3432	0.0036	0.3453	0.0045	0.3472	0.0061	0.3502	0.0071	0.3518	0.0082	0.3536	0.0104	0.3568
1	.4505	1.2377	.5024	1.2129	.5528	1.1856	.5920	1.1618	.6486	1.1234	.6758	1.1029	.7051	1.0793	.7532	1.0367
2	.4386	.5227	.4853	.4853	.5291	.4440	.5618	.4082	.6066	.3502	.6269	.3194	.6476	.2841	.6787	.2205
3	1.1407	.6586	1.2129	.5024	1.2635	.3386	1.2879	.2040	1.2971	0	1.2895	-.1015	1.2716	-.2122	1.2187	-.3960
4	.6720	.1185	.6863	0	.6802	-.1199	.6604	-.2146	.6066	-.3502	.5692	-.4136	.5203	-.4790	.4194	-.5773
5	1.2971	-.2287	1.2129	-.5024	1.0715	-.7503	.9220	-.9220	.6486	-1.1234	.4950	-1.1950	.3165	-1.2498	0	-1.2814
6	.5909	-.3412	.4853	-.4853	.3453	-.5982	.2146	-.6604	0	-.7004	-.1101	-.6949	-.2296	-.6689	-.4194	-.5773
7	.8467	-1.0090	.5024	-1.2129	.1140	-1.3031	-.2040	-1.2879	-.6486	-1.1234	-.8400	-.9836	-1.0174	-.7918	-1.2187	-.3960
8	.2334	-.6412	0	-.6863	-.2362	-.6490	-.4082	-.5618	-.6066	-.3502	-.6691	-.2174	-.7048	-.0584	-.6787	.2205
9	0	-1.3172	-.5024	-1.2129	-.9250	-.9250	-1.1618	-.5920	-1.2971	0	-1.2577	.3020	-1.1338	.6136	-.7532	1.0367
10	-.2334	-.6412	-.4853	-.4853	-.6490	-.2362	-.6944	0	-.6066	.3502	-.4975	.4975	-.3366	.6219	0	.7136
11	-.8467	-1.0090	-1.2129	-.5024	-1.3031	.1140	-1.1618	.5920	-.6486	1.1234	-.3020	1.2577	.1065	1.2848	.7532	1.0367
12	-.5909	-.3412	-.6863	0	-.5982	.3453	-.4082	.5618	0	.7004	.2174	.6691	.4344	.5581	.6787	.2205
13	-1.2971	-.2287	-1.2129	.5024	-.7503	1.0715	-.2040	1.2879	.6486	1.1234	.9836	.8400	1.2194	.4186	1.2187	-.3960
14	-.6720	.1185	-.4853	.4853	-.1199	.6802	.2146	.6604	.6066	.3502	.6949	.1101	.6855	-.1736	.4194	-.5773
15	-1.1407	.6586	-.5024	1.2129	.3386	1.2635	.9220	.9220	1.2971	0	1.1950	-.4950	.8732	-.9485	0	-1.2814
16	-.4386	.5227	0	.6863	.4440	.5291	.6604	.2146	.6066	-.3502	.4136	-.5692	.1164	-.6975	-.4194	-.5773
17	-.4505	1.2377	.5024	1.2129	1.1856	.5528	1.2879	-.2040	.6486	-1.1234	.1015	-1.2895	-.5179	-1.1806	-1.2187	-.3960
18	0	.6824	.4853	.4853	.6907	0	.5618	-.4082	0	-.7004	-.3194	-.6269	-.5920	-.3868	-.6787	.2205
19	.4505	1.2377	1.2129	.5024	1.1856	-.5528	.5920	-1.1618	-.6486	-1.1234	-1.1029	-.6758	-1.2892	0	-.7532	1.0367
20	.4386	.5227	.6863	0	.4440	-.5291	0	-.6944	-.6066	-.3502	-.7036	0	-.5920	.3868	0	.7136
21	1.1407	.6586	1.2129	-.5024	.3386	-1.2635	-.5920	-1.1618	-1.2971	0	-1.1029	.6758	-.5179	1.1806	.7532	1.0367
22	.6720	.1185	.4853	-.4853	-.1199	-.6802	-.5618	-.4082	-.6066	.3502	-.3194	.6269	.1164	.6975	.6787	.2205
23	1.2971	-.2287	.5024	-1.2129	-.7503	-1.0715	-1.2879	-.2040	-.6486	1.1234	.1015	1.2895	.8732	.9485	1.2187	-.3960
24	.5909	-.3412	0	-.6863	-.5982	-.3453	-.6604	.2146	0	.7004	.4136	.5692	.6855	.1736	.4194	-.5773
25	.8467	-1.0090	-.5024	-1.2129	-1.3031	-.1140	-.9220	.9220	.6486	1.1234	1.1950	.4950	1.2194	-.4186	0	-1.2814
26	.2334	-.6412	-.4853	-.4853	-.6490	.2362	-.2146	.6604	.6066	.3502	.6949	-.1101	.4344	-.5581	-.4194	-.5773
27	0	-1.3172	-1.2129	-.5024	-.9250	.9250	.2040	1.2879	1.2971	0	.9836	-.8400	.1065	-1.2848	-1.2187	-.3960
28	-.2334	-.6412	-.6863	0	-.2362	.6490	.4082	.5618	.6066	-.3502	.2174	-.6691	-.3366	-.6219	-.6787	.2205
29	-.8467	-1.0090	-1.2129	.5024	.1140	1.3031	1.1618	.5920	.6486	-1.1234	-.3020	-1.2577	-1.1338	-.6136	-.7532	1.0367
30	-.5909	-.3412	-.4853	.4853	.3453	.5982	.6944	0	0	-.7004	-.4975	-.4975	-.7048	.0584	0	.7136
31	-1.2971	-.2287	-.5024	1.2129	1.0715	.7503	1.1618	-.5920	-.6486	-1.1234	-1.2577	-.3020	-1.0174	.7918	.7532	1.0367
32	-.6720	.1185	0	.6863	.6802	+.1199	.4082	-.5618	-.6066	-.3502	-.6691	.2174	-.2296	.6689	.6787	.2205
33	-1.1407	.6586	.5024	1.2129	1.2635	-.3386	.2040	-1.2879	-1.2971	0	-.8400	.9838	.3165	1.2498	1.2187	-.3960
34	-.4386	.5227	.4853	.4853	.5291	-.4440	-.2146	-.6604	-.6066	.3502	-.1101	.6949	.5203	.4790	.4194	-.5773
35	-.4505	1.2377	1.2129	.5024	.5528	-1.1856	-.9220	-.9220	-.6486	1.1234	.4950	1.1950	1.2716	.2122	0	-1.2814
36	-.0019	.3412	.6863	0	.0036	-.3453	-.6604	-.2146	-.0061	.3502	.5692	.4136	.6476	-.2841	-.4194	-.5773
37	0	0	1.2129	-.5024	0	0	-1.2879	.2040	0	0	1.2895	.1015	.7051	-1.0793	-1.2187	-.3960
38	0	0	.4853	-.4853	0	0	-.5618	.4082	0	0	.6269	-.3194	.0082	-.3536	-.6787	.2205
39	0	0	.5024	-1.2129	0	0	-.5920	1.1618	0	0	.6758	-1.1029	0	0	-.7532	1.0367
40	0	0	.0026	-.3432	0	0	-.0045	.3472	0	0	.0071	-.3518	0	0	-.0104	.3568
Σ ③ × ○ =	-3.6754	-8.5426	-4.9653	-7.6728	-6.0084	-6.1534	-6.6216	-4.9584	-6.5468	-3.4560	-5.9932	-2.7532	-5.4450	-2.1104	-4.3720	-2.2040
Σ ④ × ○ =	-.47296	.24099	-.11221	.31202	-.02447	.34353	-.17157	.34490	-.07124	.31027	.14917	.27681	.16826	.23831	-.00867	.16794
Σ ⑤ × ○ =	.004880		.021457		.02344		.013288		.013069		.021180		.018536		.001826	
Σ ⑥ × ○ =	-.0016910		-.0007251		-.0003985		-.0006844		-.0003110		.0003050		.0003992		.0001287	

TABLE IV.- SUMS NEEDED FOR THE EVALUATION OF THE PARAMETERS OF EXAMPLE II BY LEAST SQUARES

	1	2	3	4	5	6	7	8	9
ν	ω_ν	$\frac{1}{\Delta\tau}\int_0^2 qy_\nu d\tau$	$\frac{1}{\Delta\tau}\int_0^2 \alpha y_\nu d\tau$	$\frac{1}{\Delta\tau}\int_0^2 \alpha^2 y_\nu d\tau$	$\frac{1}{\Delta\tau}\int_0^2 \alpha^3 y_\nu d\tau$	$\frac{1}{\omega_\nu}\Delta\tau\int_0^2 \alpha \frac{dy_\nu}{d\tau} d\tau$	$\frac{1}{\omega_\nu}\Delta\tau\int_0^2 q \frac{dy_\nu}{d\tau} d\tau$	1 × 6	1 × 7
2	1.5708	2.8046	0.90858	0.056655	0.0021576	-0.67276	-3.6366	-1.05677	-5.7124
3	2.4166	2.8806	.31917	.032479	.0005784	-.54119	-4.7790	-1.30784	-11.5489
4	3.1416	2.0718	-.58916	-.008983	-.0017933	-.39312	-5.8272	-1.23503	-18.3067
5	3.9270	1.4802	-.62799	-.007054	-.0019516	-.24078	-6.8972	-.94554	-27.0853
6	4.7124	.7873	-.20354	.017034	-.0009356	-.09202	-8.0420	-.43364	-37.8971
7	5.5440	-.6023	-.23443	.017091	-.0010623	.04498	-8.5924	.24937	-47.6363
8	6.2832	-2.3114	-.54096	.002066	-.0018572	.15638	-8.6146	.98257	-54.1273
9	6.9813	-3.6754	-.47296	.004880	-.0016910	.24099	-8.5426	1.68242	-59.6385
10	7.8540	-4.9653	-.11221	.021457	-.0007251	.31202	-7.6728	2.45061	-60.2622
11	8.7267	-6.0084	-.02447	.023436	-.0003985	.34353	-6.1534	2.99788	-53.6989
12	9.4248	-6.6216	-.17157	.013288	-.0006844	.34490	-4.9584	3.25061	-46.7319
13	10.4720	-6.5468	-.07124	.013069	-.0003110	.31027	-3.4560	3.24915	-36.1912
14	10.9956	-5.9932	.14917	.021180	.0003050	.27681	-2.7532	3.04369	-30.2731
15	11.5743	-5.4450	.16826	.018536	.0003992	.23831	-2.1104	2.75827	-24.4264
16	12.5664	-4.3720	-.00867	.001826	-.0001287	.16794	-2.2040	2.11040	-27.6963

$\Sigma\ 2^2 = 274.623$
$\Sigma\ 2 \times 3 = 4.82144$
$\Sigma\ 2 \times 4 = -.452942$
$\Sigma\ 2 \times 5 = .0206480$
$\Sigma\ 2 \times 8 = -135.099$
$\Sigma\ 2 \times 9 = 1819.62$
$\Sigma\ 3^2 = 2.37992$
$\Sigma\ 3 \times 4 = .0607352$
$\Sigma\ 3 \times 5 = .00701547$
$\Sigma\ 3 \times 8 = -1.59129$

$\Sigma\ 3 \times 9 = 105.573$
$\Sigma\ 4^2 = .00715800$
$\Sigma\ 4 \times 5 = .000100287$
$\Sigma\ 4 \times 8 = .250476$
$\Sigma\ 4 \times 9 = -6.99579$
$\Sigma\ 5^2 = .0000218456$
$\Sigma\ 5 \times 8 = -.00795442$
$\Sigma\ 5 \times 9 = .447033$
$\Sigma\ 8^2 = 66.7349$
$\Sigma\ 8 \times 9 = -875.698$

TABLE V.- CALCULATION OF THE INTEGRALS NEEDED FOR EXAMPLE III

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
n	t_n	x_n	$\Gamma_n(y_2)$	$\frac{1}{\omega_2}\Gamma_n(\dot{y}_2)$	$\frac{1}{\omega_2^2}\Gamma_n(\ddot{y}_2)$	$\frac{1}{\omega_2^3}\Gamma_n(\dddot{y}_2)$	$\frac{1}{\omega_2^4}\Gamma_n\left(y_2^{(4)}\right)$	$\Gamma_n(y_3)$	$\frac{1}{\omega_3}\Gamma_n(\dot{y}_3)$	$\frac{1}{\omega_3^2}\Gamma_n(\ddot{y}_3)$	$\frac{1}{\omega_3^3}\Gamma_n(\dddot{y}_3)$	$\frac{1}{\omega_3^4}\Gamma_n\left(y_3^{(4)}\right)$	$\Gamma_n(y_4)$	$\frac{1}{\omega_4}\Gamma_n(\dot{y}_4)$	$\frac{1}{\omega_4^2}\Gamma_n(\ddot{y}_4)$	$\frac{1}{\omega_4^3}\Gamma_n(\dddot{y}_4)$	$\frac{1}{\omega_4^4}\Gamma_n\left(y_4^{(4)}\right)$	$\Gamma_n(y_5)$	$\frac{1}{\omega_5}\Gamma_n(\dot{y}_5)$	$\frac{1}{\omega_5^2}\Gamma_n(\ddot{y}_5)$
0	0	0.2500	0	-0.0005	-0.0095	0.0102	8.1916	-0.0001	-0.0018	-0.0215	0.0363	8.4352	-0.0003	-0.0039	-0.0341	0.0779	8.6994	-0.0006	-0.0072	-0.0489
1	.05	.1055	.0001	.0041	.1161	2.4315	29.6700	.0006	.0146	.2676	3.5733	26.6079	.0018	.0314	.4378	4.3938	23.1296	.0043	.0595	.6541
2	.10	-.0804	.0001	.0073	.1749	2.3669	12.4930	.0009	.0271	.4037	3.3274	7.8712	.0027	.0600	.6584	3.8317	2.6587	.0073	.1177	.9706
3	.15	-.2795	.0045	.0698	.8181	6.1293	15.3516	.0235	.2325	1.6704	6.8276	-2.9138	.0625	.4587	2.3440	5.4774	-18.8795	.1381	.7700	2.7767
4	.20	-.4562	.0053	.0707	.6640	3.5945	2.3829	.0276	.2297	1.2596	2.9847	-10.9597	.0729	.4365	1.5745	.7706	-20.1533	.1581	.6863	1.4704
5	.25	-.5763	.0298	.2804	1.8810	6.7998	-7.5238	.1431	.8076	2.8899	1.9518	-34.9570	.3398	1.3202	2.5629	-5.2809	-41.0058	.6405	1.6559	.7670
6	.30	-.6159	.0269	.2189	1.2075	3.0017	-9.8691	.1249	.5862	1.5137	-1.3455	-22.2682	.2828	.8559	.7347	-5.9403	-16.7939	.4927	.8529	-.9706
7	.35	-.5678	.1011	.6510	2.7597	3.8043	-29.6628	.4218	1.4717	2.2096	-7.7572	-39.1889	.8385	1.6774	-.7600	-14.0220	-6.4635	1.2172	.9454	-4.1977
8	.40	-.4438	.0778	.4375	1.5134	.6464	-19.2463	.3045	.8612	.5634	-6.2131	-14.7526	.5524	.7387	-1.5404	-7.0267	11.4538	.6851	0	-2.8431
9	.45	-.2718	.2390	1.1097	2.9273	-1.9909	-41.8438	.8188	1.6998	-.5823	-14.1061	-8.7553	1.2574	.7846	-4.5846	-7.6575	43.2193	1.2172	-.9454	-4.1977
10	.50	-.0895	.1651	.6699	1.3589	-2.6797	-21.7427	.5129	.7925	-1.1772	-7.3143	6.7216	.6789	0	-2.7861	0	28.2702	.4927	-.8529	-.9706
11	.55	.0658	.4471	1.5177	2.0951	-8.5184	-38.5148	1.1793	1.1520	-3.8272	-11.0392	33.0658	1.2574	-.7846	-4.5846	7.6575	43.2193	.6405	-1.6559	.7670
12	.60	.1652	.2848	.8368	.6853	-5.7436	-15.9340	.6544	.3230	-2.5416	-3.2323	24.9525	.5524	-.7387	-1.5404	7.0267	11.4538	.1581	-.6863	1.4704
13	.65	.1960	.7049	1.7191	.3444	-13.2850	-20.0016	1.3256	0	-5.2560	0	52.2828	.8385	-1.6774	-.7600	14.0220	-6.4635	.1381	-.7700	2.7767
14	.70	.1639	.4206	.8651	-.3676	-7.3376	-3.5686	.6544	-.3230	-2.5416	3.2323	24.9525	.2828	-.8559	.7347	5.9403	-16.7938	.0073	-.1177	.9706
15	.75	.0904	.9702	1.6006	-1.8810	-14.3237	7.5238	1.1793	-1.1520	-3.8272	11.0392	33.0658	.3398	-1.3202	2.5629	5.2809	-41.0058	.0043	-.0595	.6541
16	.80	.0071	.5472	.7169	-1.5039	-6.7447	11.0547	.5129	-.7925	-1.1772	7.3143	6.7216	.0729	-.4365	1.5745	-.7706	-20.1533	-.0013	0	-.0978
17	.85	-.0540	1.1895	1.1379	-3.9222	-10.9599	34.3128	.8188	-1.6998	-.5823	14.1061	-8.7553	.0625	-.4587	2.3440	-5.4774	-18.8795	.0043	.0595	.6541
18	.90	-.0689	.6373	.4067	-2.3737	-4.0231	22.6874	.3045	-.8612	.5634	6.2131	-14.7526	.0027	-.0600	.6584	-3.8317	2.6587	.0073	.1177	.9706
19	.95	-.0285	1.3138	.4121	-5.1385	-4.0960	50.6886	.4218	-1.4717	2.2096	7.7572	-39.1889	.0018	-.0314	.4378	-4.3938	23.1296	.1381	.7700	2.7767
20	1.00	.0590	.6699	0	-2.6988	0	27.1021	.1249	-.5862	1.5137	1.3455	-22.2682	-.0005	0	-.0682	0	17.3989	.1581	.6863	1.4704
21	1.05	.1701	1.3138	-.4121	-5.1385	4.0960	50.6886	.1431	-.8076	2.8899	-1.9518	-34.9570	.0018	.0314	.4378	4.3938	23.1296	.6405	1.6559	.7670
22	1.10	.2733	.6373	-.4067	-2.3737	4.0231	22.6874	.0276	-.2297	1.2596	-2.9847	-10.9597	.0027	.0600	.6584	3.8317	2.6587	.4927	.8529	-.9706
23	1.15	.3369	1.1895	-1.1379	-3.9222	10.9599	34.3128	.0235	-.2325	1.6704	-6.8276	-2.9138	.0625	.4587	2.3440	5.4774	-18.8795	1.2172	.9454	-4.1977
24	1.20	.3385	.5472	-.7169	-1.5039	6.7447	11.0547	.0009	-.0271	.4037	-3.3274	7.8712	.0729	.4365	1.5745	.7706	-20.1533	.6851	0	-2.8431
25	1.25	.2713	.9702	-1.6006	-1.8810	14.3237	7.5238	.0006	-.0146	.2676	-3.5733	26.6079	.3398	1.3202	2.5629	-5.2809	-41.0058	1.2172	-.9454	-4.1977
26	1.30	.1463	.4206	-.8651	-.3676	7.3376	-3.5686	-.0001	-.0018	-.0215	-.0363	8.4352	.2828	.8559	.7347	-5.9403	-16.7938	.4927	-.8529	-.9706
27	1.35	-.0102	.7049	-1.7191	.3444	13.2850	-20.0016	0	0	0	0	0	.8385	1.6774	-.7600	-14.0220	-6.4635	.6405	-1.6559	.7670
28	1.40	-.1631	.2848	-.8368	.6858	5.7436	-15.9340	0	0	0	0	0	.5524	.7387	-1.5404	-7.0267	11.4538	.1581	-.6863	1.4704
29	1.45	-.2778	.4471	-1.5177	2.0951	8.5184	-38.5148	0	0	0	0	0	1.2574	.7846	-4.5846	-7.6575	43.2193	.1381	-.7700	2.7767
30	1.50	-.3289	.1651	-.6699	1.3589	2.6797	-21.7427	0	0	0	0	0	.6789	0	-2.7861	0	28.2702	.0073	-.1177	.9706
31	1.55	-.3070	.2390	-1.1097	2.9273	1.9909	-41.8438	0	0	0	0	0	1.2574	-.7846	-4.5846	7.6575	43.2193	.0043	-.0595	.6541
32	1.60	-.2210	.0778	-.4375	1.5134	-.6464	-19.2463	0	0	0	0	0	.5524	-.7387	-1.5404	7.0267	11.4538	-.0006	.0072	-.0489
33	1.65	-.0958	.1011	-.6510	2.7597	-3.8043	-29.6628	0	0	0	0	0	.8385	-1.6774	-.7600	14.0220	-6.4635	0	0	0
34	1.70	.0348	.0269	-.2189	1.2075	-3.0017	-9.8691	0	0	0	0	0	.2828	-.8559	.7347	5.9403	-16.7939	0	0	0
35	1.75	.1362	.0298	-.2804	1.8810	-6.7998	-7.5238	0	0	0	0	0	.3398	-1.3202	2.5629	5.2809	-41.0058	0	0	0
36	1.80	.1832	.0053	-.0707	.6640	-3.5945	2.3829	0	0	0	0	0	.0729	-.4365	1.5745	-.7706	-20.1533	0	0	0
37	1.85	.1655	.0045	-.0698	.8181	-6.1293	15.3516	0	0	0	0	0	.0625	-.4587	2.3440	-5.4774	-18.8795	0	0	0
38	1.90	.0911	.0001	-.0073	.1749	-2.3669	12.4930	0	0	0	0	0	.0027	-.0600	.6584	-3.8317	2.6587	0	0	0
39	1.95	-.0166	.0001	-.0041	.1161	-2.4315	29.6700	0	0	0	0	0	.0018	-.0314	.4378	-4.3938	23.1296	0	0	0
40	2.00	-.1250	0	.0005	-.0095	-.0102	8.1916	0	0	0	0	0	-.0003	.0039	-.0341	-.0779	8.6994	0	0	0
		Σ ③ × ○ =	1.0427	-.6518	-11.068	-10.961	109.208	.1882	-2.9145	-5.7390	1.642	100.693	-2.1316	-3.2009	6.5197	24.080	-39.379	-.8385	-1.7570	-.7993

○ Denotes the column in which the numbers appear.

TABLE V.- CALCULATION OF THE INTEGRALS NEEDED FOR EXAMPLE III - Continued

1	22	23	24	25	26	27	28	29	30
n	$\frac{1}{\omega_5^3}\Gamma_n(\dddot{y}_5)$	$\frac{1}{\omega_5^4}\Gamma_n\left(y_5^{(4)}\right)$	$\Gamma_n(y_6)$	$\frac{1}{\omega_6}\Gamma_n(\dot{y}_6)$	$\frac{1}{\omega_6^2}\Gamma_n(\ddot{y}_6)$	$\frac{1}{\omega_6^3}\Gamma_n(\dddot{y}_6)$	$\frac{1}{\omega_6^4}\Gamma_n\left(y_6^{(4)}\right)$	$\Gamma_n(y_7)$	$\frac{1}{\omega_7}\Gamma_n(\dot{y}_7)$
0	0.1472	9.0182	-.0013	-.0119	-.0630	0.2440	9.3408	-0.0023	-0.0181
1	5.0777	18.6458	.0086	.0990	.8911	5.5190	13.6260	.0160	.1540
2	3.9405	-3.8825	.0167	.2016	1.2845	3.5155	-10.7551	.0344	.3194
3	2.2356	-32.3688	.2531	1.0996	2.7259	-2.1391	-38.7183	.4151	1.3954
4	-2.7453	-23.5271	.2807	.8830	.7963	-6.2033	-17.8915	.4355	.9366
5	-11.9391	-24.3300	.9610	1.5313	-1.8441	-13.4369	7.3762	1.2114	.8633
6	-7.8230	3.8825	.6568	.4372	-2.5464	-4.4199	24.8918	.6869	-.2701
7	-9.0970	38.0529	1.2888	-.3923	-4.8937	3.8285	47.3897	.9913	-1.4567
8	0	29.0177	.5602	-.7683	-1.5917	7.3954	11.9841	.2635	-.8824
9	9.0970	38.0529	.7065	-1.6548	.2486	12.5342	-18.1848	.1981	-.9338
10	7.8230	3.8825	.1553	-.6944	1.5148	2.7777	-24.2370	-.0012	-.0865
11	11.9391	-24.3300	.1145	-.6689	2.6165	-3.2396	-27.6566	.0017	-.0251
12	2.7453	-23.5271	-.0016	-.0480	.6555	-4.1300	2.9952	-.0063	-.0081
13	-2.2356	-32.3688	.0009	-.0157	.2579	-2.1966	26.7833	.0702	.4659
14	-3.9405	-3.8825	-.0036	-.0080	.0954	2.7032	14.3246	.1218	.6296
15	-5.0777	18.6458	.0390	.3127	1.8441	6.0606	-7.3763	.6995	1.6267
16	0	18.0364	.0661	.4379	1.6547	.9198	-21.3249	.5928	.7194
17	5.0777	18.6458	.4571	1.4762	1.9099	-8.1642	-35.4546	1.2935	0
18	3.9405	-3.8825	.4248	.9220	-.3483	-8.0099	-4.2242	.5928	-.7194
19	2.2356	-32.3688	1.1705	1.0849	-3.7562	-10.2549	32.2153	.6995	-1.6267
20	-2.7453	-23.5271	.6919	0	-2.9037	0	29.7925	.1218	-.6296
21	-11.9391	-24.3300	1.1705	-1.0849	-3.7562	10.2549	32.2153	.0702	-.4659
22	-7.8230	3.8825	.4248	-.9220	-.3483	8.0099	-4.2242	-.0063	.0081
23	-9.0970	38.0529	.4571	-1.4762	1.9099	8.1642	-35.4546	.0017	.0251
24	0	29.0177	.0661	-.4379	1.6547	-.9198	-21.3249	-.0012	.0865
25	9.0970	38.0529	.0390	-.3127	1.8441	-6.0606	-7.3763	.1981	.9338
26	7.8230	3.8825	-.0036	.0080	.0954	-2.7032	14.3246	.2635	.8824
27	11.9391	-24.3300	.0009	.0157	.2579	2.1966	26.7833	.9913	1.4567
28	2.7453	-23.5271	-.0016	.0480	.6555	4.1300	2.9952	.6869	.2701
29	-2.2356	-32.3688	.1145	.6689	2.6165	3.2396	-27.6566	1.2114	-.8633
30	-3.9405	-3.8825	.1553	.6944	1.5148	-2.7777	-24.2370	.4355	-.9366
31	-5.0777	18.6458	.7065	1.6548	.2486	-12.5342	-18.1848	.4151	-1.3954
32	-.1472	9.0182	.5602	.7683	-1.5917	-7.3954	11.9841	.0344	-.3194
33	0	0	1.2888	.3923	-4.8937	-3.8285	47.3897	.0160	-.1540
34	0	0	.6568	-.4372	-2.5464	4.4199	24.8918	-.0023	.0181
35	0	0	.9610	-1.5313	-1.8441	13.4369	7.3762	0	0
36	0	0	.2807	-.8830	.7963	6.2033	-17.8915	0	0
37	0	0	.2531	-1.0996	2.7259	2.1391	-38.7183	0	0
38	0	0	.0167	-.2016	1.2845	-3.5155	-10.7551	0	0
39	0	0	.0086	-.0990	.8911	-5.5190	13.6260	0	0
40	0	0	-.0013	.0119	-.0630	-.2440	9.3408	0	0
Σ (3) × O =	12.817	16.004	-2.1764	-3.8832	5.9052	18.875	-24.220	-2.8302	2.0668

1	31	32	33	34	35	36	37	38
n	$\frac{1}{\omega_7^2}\Gamma_n(\ddot{y}_7)$	$\frac{1}{\omega_7^3}\Gamma_n(\dddot{y}_7)$	$\frac{1}{\omega_7^4}\Gamma_n\left(y_7^{(4)}\right)$	$\Gamma_n(y_8)$	$\frac{1}{\omega_8}\Gamma_n(\dot{y}_8)$	$\frac{1}{\omega_8^2}\Gamma_n(\ddot{y}_8)$	$\frac{1}{\omega_8^3}\Gamma_n(\dddot{y}_8)$	$\frac{1}{\omega_8^4}\Gamma_n\left(y_8^{(4)}\right)$
0	-0.0750	0.3772	9.6501	-0.0037	-0.0247	-0.0815	0.5204	9.8665
1	1.1495	5.7026	7.9871	.0255	.2138	1.3724	5.6176	2.9231
2	1.5668	2.4530	-17.5367	.0593	.4450	1.7274	1.0239	-22.3468
3	2.0743	-6.8785	-36.0510	.5833	1.5521	1.0434	-10.2087	-26.1979
4	-.4222	-8.2431	-3.4286	.5694	.7975	-1.6460	-7.7268	12.4803
5	-4.1498	-8.2086	37.4650	1.2825	0	-4.8315	0	46.5495
6	-2.8374	2.7693	28.7870	.5694	-.7975	-1.6460	7.7268	12.4803
7	-2.1279	12.8552	11.1167	.5833	-1.5521	1.0434	10.2087	-26.1979
8	.9395	5.9905	-19.6573	.0593	-.4450	1.7274	-1.0239	-22.3468
9	2.7081	.0192	-34.8378	.0255	-.2138	1.3724	-5.6176	2.9231
10	.8970	-4.3092	-1.8624	-.0074	0	-.1629	0	19.7330
11	.3476	-2.4496	24.9378	.0255	.2138	1.3724	5.6176	2.9231
12	.1616	3.1819	13.1732	.0593	.4450	1.7274	1.0239	-22.3468
13	2.2021	4.9256	-16.5351	.5833	1.5521	1.0434	-10.2087	-26.1979
14	1.6617	-1.6592	-25.0287	.5694	.7975	-1.6460	-7.7268	12.4803
15	.2654	-12.1397	-18.0769	1.2825	0	-4.8315	0	46.5495
16	-1.8920	-7.0626	15.9033	.5694	-.7975	-1.6460	7.7268	12.4803
17	-4.9385	0	47.9886	.5833	-1.5521	1.0434	10.2087	-26.1979
18	-1.8920	7.0626	15.9033	.0593	-.4450	1.7274	-1.0239	-22.3468
19	.2654	12.1397	-18.0769	.0255	-.2138	1.3724	-5.6176	2.9231
20	1.6617	1.6592	-25.0287	-.0074	0	-.1629	0	19.7330
21	2.2021	-4.9256	-16.5351	.0255	.2138	1.3724	5.6176	2.9231
22	.1616	-3.1819	13.1732	.0593	.4450	1.7274	1.0239	-22.3468
23	.3476	2.4496	24.9378	.5833	1.5521	1.0434	-10.2087	-26.1979
24	.8970	4.3092	-1.8624	.5694	.7975	-1.6460	-7.7268	12.4803
25	2.7081	-.0192	-34.8378	1.2825	0	-4.8315	0	46.5495
26	.9395	-5.9905	-19.6573	.5694	-.7975	-1.6460	7.7268	12.4803
27	-2.1279	-12.8552	11.1167	.5833	-1.5521	1.0434	10.2087	-26.1979
28	-2.8374	-2.7693	28.7870	.0593	-.4450	1.7274	-1.0239	-22.3468
29	-4.1498	8.2086	37.4650	.0255	-.2138	1.3724	-5.6176	2.9231
30	-.4222	8.2431	-3.4286	-.0074	0	-.1629	0	19.7330
31	2.0743	6.8785	-36.0510	.0255	.2138	1.3724	5.6176	2.9231
32	1.5668	-2.4530	-17.5367	.0593	.4450	1.7274	1.0239	-22.3468
33	1.1495	-5.7026	7.9871	.5833	1.5521	1.0434	-10.2087	-26.1979
34	-.0750	-.3772	9.6501	.5694	.7975	-1.6460	-7.7268	12.4803
35	0	0	0	1.2825	0	-4.8315	0	46.5495
36	0	0	0	.5694	-.7975	-1.6460	7.7268	12.4803
37	0	0	0	.5833	-1.5521	1.0434	10.2087	-26.1979
38	0	0	0	.0593	-.4450	1.7274	-1.0239	-22.3468
39	0	0	0	.0255	-.2138	1.3724	-5.6176	2.9231
40	0	0	0	-.0037	.0247	-.0815	-.5204	9.8665
Σ (3) × O =	7.3470	-6.9452	-24.007	-.4323	1.0724	-1.1109	-4.687	4.474

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TABLE V.- CALCULATION OF THE INTEGRALS NEEDED FOR EXAMPLE III - Continued

1	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55
n	$\Gamma_n(y_9)$	$\frac{1}{\omega_9}\Gamma_n(\dot{y}_9)$	$\frac{1}{\omega_9^2}\Gamma_n(\ddot{y}_9)$	$\frac{1}{\omega_9^3}\Gamma_n(\dddot{y}_9)$	$\frac{1}{\omega_9^4}\Gamma_n\left(y_9^{(4)}\right)$	$\Gamma_n(y_{10})$	$\frac{1}{\omega_{10}}\Gamma_n(\dot{y}_{10})$	$\frac{1}{\omega_{10}^2}\Gamma_n(\ddot{y}_{10})$	$\frac{1}{\omega_{10}^3}\Gamma_n(\dddot{y}_{10})$	$\frac{1}{\omega_{10}^4}\Gamma_n\left(y_{10}^{(4)}\right)$	$\Gamma_n(y_{11})$	$\frac{1}{\omega_{11}}\Gamma_n(\dot{y}_{11})$	$\frac{1}{\omega_{11}^2}\Gamma_n(\ddot{y}_{11})$	$\frac{1}{\omega_{11}^3}\Gamma_n(\dddot{y}_{11})$	$\frac{1}{\omega_{11}^4}\Gamma_n\left(y_{11}^{(4)}\right)$	$\Gamma_n(y_{12})$	$\frac{1}{\omega_{12}}\Gamma_n(\dot{y}_{12})$
0	-0.0054	-0.0316	-0.0825	0.6743	9.9963	-0.0081	-0.0406	-0.0754	0.8859	10.0283	-0.0116	-0.0495	-0.0574	1.1099	9.8899	-0.0148	-0.0561
1	.0374	.2789	1.5661	5.3302	-1.7182	.0570	.3699	1.7718	4.7127	-7.0873	.0822	.4686	1.9236	3.8511	-11.7325	.1065	.5503
2	.0926	.5743	1.7650	-.6440	-25.2264	.1487	.7352	1.6211	-2.9409	-25.9382	.2204	.8739	1.2533	-5.1477	-23.1685	.2869	.9532
3	.7492	1.5717	-.1786	-11.9552	-12.3758	.9430	1.4019	-1.7718	-11.8001	7.0873	1.0951	1.0368	-3.1072	-9.1821	24.0563	1.1721	.6457
4	.6668	.5062	-2.5929	-5.1313	25.1812	.7189	0	-3.0916	0	31.8200	.6783	-.5144	-2.6384	5.1625	25.3541	.5870	-.8219
5	1.2009	-.7848	-4.0571	7.3462	36.2838	.9430	-1.4019	-1.7718	11.8001	7.0873	.6183	-1.4715	.6401	9.6229	-20.3745	.3855	-1.2186
6	.3804	-.9749	.0825	8.0853	-9.9963	.1487	-.7352	1.6211	2.9409	-25.9382	.0128	-.2989	1.5526	-2.9875	-15.8708	-.0258	-.0685
7	.2616	-1.0636	2.4910	2.0160	-34.5657	.0570	-.3699	1.7718	-4.7127	-7.0873	.0059	-.0487	.6105	-1.7937	19.4104	.0131	.1074
8	-.0094	-.0681	.8279	-4.4873	.0452	-.0163	0	-.1507	0	20.0565	-.0189	.0842	.8658	4.4201	-.1083	.0529	.4875
9	.0016	0	.3572	0	24.7515	.0570	.3699	1.7718	4.7127	-7.0873	.3789	1.2345	1.9379	-4.9381	-31.0066	.8228	1.4214
10	-.0094	.0681	.8279	4.4873	.0452	.1487	.7352	1.6211	-2.9409	-25.9382	.5538	.8770	-1.4250	-8.2646	9.0549	.7277	0
11	.2616	1.0636	2.4910	-2.0160	-34.5657	.9430	1.4019	-1.7718	-11.8001	7.0873	1.2217	-.3801	-4.2526	3.5087	38.8628	.8228	-1.4214
12	.3804	.9749	.0825	-8.0853	-9.9963	.7189	0	-3.0916	0	31.8200	.3866	-.9960	.0574	8.1671	-9.8899	.0529	-.4875
13	1.2009	.7848	-4.0571	-7.3462	36.2838	.9430	-1.4019	-1.7718	11.8001	7.0873	.1961	-.8487	2.3290	-.3424	-27.1303	.0131	-.1074
14	.6668	-.5062	-2.5929	5.1313	25.1812	.1487	-.7352	1.6211	2.9409	-25.9382	-.0242	.0030	.1717	-3.1169	14.1136	-.0258	.0685
15	.7492	-1.5717	-.1786	11.9552	-12.3758	.0570	-.3699	1.7718	-4.7127	-7.0873	.0260	.1977	1.1693	4.2441	6.9503	.3855	1.2186
16	.0926	-.5743	1.7650	.6440	-25.2264	-.0163	0	-.1507	0	20.0565	.0905	.5986	1.7726	-.7423	-25.2457	.5870	.8219
17	.0374	-.2789	1.5661	-5.3302	-1.7182	.0570	.3699	1.7718	4.7127	-7.0873	.8757	1.4228	-1.2505	-11.4167	.9640	1.1721	-.6457
18	-.0108	0	-.1650	0	19.9926	.1487	.7352	1.6211	-2.9409	-25.9382	.7244	0	-3.1053	0	31.7416	.2869	-.9532
19	.0374	.2789	1.5661	5.3302	-1.7182	.9430	1.4019	-1.7718	-11.8001	7.0873	.8757	-1.4228	-1.2505	11.4167	.9640	.1065	-.5503
20	.0926	.5743	1.7650	-.6440	-25.2264	.7189	0	-3.0916	0	31.8200	.0905	-.5986	1.7726	.7423	-25.2457	-.0297	0
21	.7492	1.5717	-.1786	-11.9552	-12.3758	.9430	-1.4019	-1.7718	11.8001	7.0873	.0260	-.1977	1.1693	-4.2441	6.9503	.1065	.5503
22	.6668	.5062	-2.5929	-5.1313	25.1812	.1487	-.7352	1.6211	2.9409	-25.9382	-.0242	-.0030	.1717	3.1169	14.1136	.2869	.9532
23	1.2009	-.7848	-4.0571	7.3462	36.2838	.0570	-.3699	1.7718	-4.7127	-7.0873	.1961	.8487	2.3290	.3424	-27.1303	1.1721	.6457
24	.3804	-.9749	.0825	8.0853	-9.9963	-.0163	0	-.1507	0	20.0565	.3866	.9960	.0574	-8.1671	-9.8899	.5870	-.8219
25	.2616	-1.0636	2.4910	2.0160	-34.5657	.0570	.3699	1.7718	4.7127	-7.0873	1.2217	.3801	-4.2526	-3.5087	38.8628	.3855	-1.2186
26	-.0094	-.0681	.8279	-4.4873	.0452	.1487	.7352	1.6211	-2.9409	-25.9382	.5538	-.8770	-1.4250	8.2646	9.0549	-.0258	-.0685
27	.0016	0	.3572	0	24.7515	.9430	1.4019	-1.7718	-11.8001	7.0873	.3789	-1.2345	1.9379	4.9381	-31.0066	.0131	.1074
28	-.0094	.0681	.8279	4.4873	.0452	.7189	0	-3.0916	0	31.8200	-.0189	-.0842	.8658	-4.4201	-.1083	.0529	.4875
29	.2616	1.0636	2.4910	-2.0160	-34.5657	.9430	-1.4019	-1.7718	11.8001	7.0873	.0059	.0487	.6105	1.7937	19.4104	.8228	1.4214
30	.3804	.9749	.0825	-8.0853	-9.9963	.1487	-.7352	1.6211	2.9409	-25.9382	.0128	.2989	1.5526	2.9875	-15.8708	.7277	0
31	1.2009	.7848	-4.0571	-7.3462	36.2838	.0570	-.3699	1.7718	-4.7127	-7.0873	.6183	1.4715	.6401	-9.6229	-20.3745	.8228	-1.4214
32	.6668	-.5062	-2.5929	5.1313	25.1812	-.0163	0	-.1507	0	20.0565	.6783	.5144	-2.6384	-5.1625	25.3541	.0529	-.4875
33	.7492	-1.5717	-.1786	11.9552	-12.3758	.0570	.3699	1.7718	4.7127	-7.0873	1.0951	-1.0368	-3.1072	9.1821	24.0563	.0131	-.1074
34	.0926	-.5743	1.7650	.6440	-25.2264	.1487	.7352	1.6211	-2.9409	-25.9382	.2204	-.8739	1.2533	5.1477	-23.1685	-.0258	.0685
35	.0374	-.2789	1.5661	-5.3302	-1.7182	.9430	1.4019	-1.7718	-11.8001	7.0873	.0822	-.4686	1.9236	-3.8511	-11.7325	.3855	1.2186
36	-.0054	.0316	-.0825	-.6743	9.9963	.7189	0	-3.0916	0	31.8200	-.0116	.0495	-.0574	-1.1099	9.8899	.5870	.8219
37	0	0	0	0	0	.9430	-1.4019	-1.7718	11.8001	7.0873	0	0	0	0	0	1.1721	-.6457
38	0	0	0	0	0	.1487	-.7352	1.6211	2.9409	-25.9382	0	0	0	0	0	.2869	-.9532
39	0	0	0	0	0	.0570	-.3699	1.7718	-4.7127	-7.0873	0	0	0	0	0	.1065	-.5503
40	0	0	0	0	0	-.0081	.0406	-.0754	-.8859	10.0283	0	0	0	0	0	-.0148	.0561
Σ ③ × ○ =	-.9567	.0375	.4663	.116	-.193	-.6614	.6929	-.1993	-1.181	.906	-.9016	.0418	.2848	.052	-.084	-.6965	.4501

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TABLE V.- CALCULATION OF THE INTEGRALS NEEDED FOR EXAMPLE III - Continued

1	56	57	58	59	60	61	62	63
n	$\frac{1}{\omega_{12}^2}\Gamma_n(\ddot{y}_{12})$	$\frac{1}{\omega_{12}^3}\Gamma_n(\dddot{y}_{12})$	$\frac{1}{\omega_{12}^4}\Gamma_n\left(y_{12}^{(4)}\right)$	$\Gamma_n(y_{13})$	$\frac{1}{\omega_{13}}\Gamma_n(\dot{y}_{13})$	$\frac{1}{\omega_{13}^2}\Gamma_n(\ddot{y}_{13})$	$\frac{1}{\omega_{13}^3}\Gamma_n(\dddot{y}_{13})$	$\frac{1}{\omega_{13}^4}\Gamma_n\left(y_{13}^{(4)}\right)$
0	-0.0348	1.2905	9.6452	-0.0205	-0.0641	0.0130	1.5494	9.0495
1	1.9986	3.0252	-14.7871	.1498	.6722	2.0257	1.6395	-18.0991
2	.8137	-6.6064	-18.6358	.3955	.9965	-.0130	-7.9272	-9.0495
3	-3.7999	-5.8131	32.9831	1.2005	0	-4.0515	0	36.1981
4	-1.7151	7.8080	12.7355	.3955	-.9965	-.0130	7.9272	-9.0495
5	1.8318	4.8744	-29.3090	.1498	-.6722	2.0257	-1.6395	-18.0991
6	.7359	-4.2465	2.9314	-.0409	0	.0260	0	18.0991
7	.8359	2.8006	14.4399	.1498	.6722	2.0257	1.6395	-18.0991
8	1.7499	.8437	-22.3807	.3955	.9965	-.0130	-7.9272	-9.0495
9	-.8665	-10.9121	-3.3268	1.2005	0	-4.0515	0	36.1981
10	-3.0992	0	31.4089	.3955	-.9965	-.0130	7.9272	-9.0495
11	-.8665	10.9121	-3.3268	.1498	-.6722	2.0257	-1.6395	-18.0991
12	1.7499	-.8437	-22.3807	-.0409	0	.0260	0	18.0991
13	.8359	-2.8006	14.4399	.1498	.6722	2.0257	1.6395	-18.0991
14	.7359	4.2465	2.9314	.3955	.9965	-.0130	-7.9272	-9.0495
15	1.8318	-4.8744	-29.3090	1.2005	0	-4.0515	0	36.1981
16	-1.7151	-7.8080	12.7355	.3955	-.9965	-.0130	7.9272	-9.0495
17	-3.7999	5.8131	32.9831	.1498	-.6722	2.0257	-1.6395	-18.0991
18	.8137	6.6064	-18.6358	-.0409	0	.0260	0	18.0991
19	1.9986	-3.0252	-14.7871	.1498	.6722	2.0257	1.6395	-18.0991
20	-.0695	0	19.2903	.3955	.9965	-.0130	-7.9272	-9.0495
21	1.9986	3.0252	-14.7871	1.2005	0	-4.0515	0	36.1981
22	.8137	-6.6064	-18.6358	.3955	-.9965	-.0130	7.9272	-9.0495
23	-3.7999	-5.8131	32.9831	.1498	-.6722	2.0257	-1.6395	-18.0991
24	-1.7151	7.8080	12.7355	-.0409	0	.0260	0	18.0991
25	1.8318	4.8744	-29.3090	.1498	.6722	2.0257	1.6395	-18.0991
26	.7359	-4.2465	2.9314	.3955	.9965	-.0130	-7.9272	-9.0495
27	.8359	2.8006	14.4399	1.2005	0	-4.0515	0	36.1981
28	1.7499	.8437	-22.3807	.3955	-.9965	-.0130	7.9272	-9.0495
29	-.8665	-10.9121	-3.3268	.1498	-.6722	2.0257	-1.6395	-18.0991
30	-3.0992	0	31.4089	-.0409	0	.0260	0	18.0991
31	-.8665	10.9121	-3.3268	.1498	.6722	2.0257	1.6395	-18.0991
32	1.7499	-.8437	-22.3807	.3955	.9965	-.0130	-7.9272	-9.0495
33	.8359	-2.8006	14.4399	1.2005	0	-4.0515	0	36.1981
34	.7359	4.2465	2.9314	.3955	-.9965	-.0130	7.9272	-9.0495
35	1.8318	-4.8744	-29.3090	.1498	-.6722	2.0257	-1.6395	-18.0991
36	-1.7151	-7.8080	12.7355	-.0205	.0641	.0130	-1.5494	9.0495
37	-3.7999	5.8131	32.9831	0	0	0	0	0
38	.8137	6.6064	-18.6358	0	0	0	0	0
39	1.9986	-3.0252	-14.7871	0	0	0	0	0
40	-.0348	-1.2905	9.6452	0	0	0	0	0
Σ ③ × ○	= -.0592	-.516	.348	-.8736	.0393	.1926	.027	-.045

1	64	65	66	67	68	69	70	71
n	$\Gamma_n(y_{14})$	$\frac{1}{\omega_{14}}\Gamma_n(\dot{y}_{14})$	$\frac{1}{\omega_{14}^2}\Gamma_n(\ddot{y}_{14})$	$\frac{1}{\omega_{14}^3}\Gamma_n(\dddot{y}_{14})$	$\frac{1}{\omega_{14}^4}\Gamma_n\left(y_{14}^{(4)}\right)$	$\Gamma_n(y_{15})$	$\frac{1}{\omega_{15}}\Gamma_n(\dot{y}_{15})$	$\frac{1}{\omega_{15}^2}\Gamma_n(\ddot{y}_{15})$
0	-0.0236	-0.0670	0.0428	1.6679	8.6524	-0.0271	-0.0690	0.0797
1	.1743	.7307	1.9997	.9141	-19.1493	.2035	.7918	1.9406
2	.4503	.9791	-.4602	-8.1122	-3.6124	.5085	.9288	-.9467
3	1.1762	-.3066	-3.8324	2.6918	33.3717	1.1233	-.6058	-3.3724
4	.2959	-.9561	.7089	6.4174	-17.1120	.1955	-.8402	1.2614
5	.0832	-.4384	1.6673	-2.9890	-6.6691	.0393	-.2426	1.2332
6	-.0456	.0243	.2902	2.4958	13.1157	-.0380	.1403	.8925
7	.3138	1.0423	1.8724	-2.7036	-25.5316	.5611	1.3042	.7356
8	.5931	.8070	-1.7133	-7.4237	12.3065	.7204	.2437	-2.8821
9	1.0833	-.8550	-3.0301	7.2568	23.2711	.7808	-1.3138	-.6755
10	.1581	-.7781	1.4706	3.1122	-23.5292	0	-.3342	1.4250
11	.0329	-.2155	1.1717	-2.9750	6.4640	.0142	.0708	.8892
12	-.0363	.1077	.8045	3.7651	2.2353	.0761	.5930	1.6256
13	.4982	1.2866	1.1347	-6.6118	-22.5808	.9817	1.0744	-2.2003
14	.6943	.4565	-2.6697	-4.4193	24.9553	.6433	-.6687	-2.1518
15	.9168	-1.2289	-1.6673	9.6581	6.6691	.3604	-1.0979	1.6575
16	.0554	-.5239	1.6706	-.4977	-20.9590	-.0516	-.0383	.3718
17	.0118	-.0623	.8253	-1.2164	14.7408	.0985	.4855	1.6981
18	-.0071	.2798	1.3691	3.0766	-10.9295	.3479	.9781	.3246
19	.7095	1.3702	-.1412	-9.3176	-10.5858	1.1744	0	-3.8117
20	.7309	0	-3.0267	0	29.7536	.3479	-.9781	.3246
21	.7095	-1.3702	-.1412	9.3176	-10.5858	.0985	-.4855	1.6981
22	-.0071	-.2798	1.3691	-3.0766	-10.9295	-.0516	.0383	.3718
23	.0118	.0623	.8253	1.2164	14.7408	.3604	1.0979	1.6575
24	.0554	.5239	1.6706	.4977	-20.9590	.6433	.6687	-2.1518
25	.9168	1.2289	-1.6673	-9.6581	6.6691	.9817	-1.0744	-2.2003
26	.6943	-.4565	-2.6697	4.4193	24.9553	.0761	-.5930	1.6256
27	.4982	-1.2866	1.1347	6.6118	-22.5808	.0142	-.0708	.8892
28	-.0363	-.1077	.8045	-3.7651	2.2353	0	.3342	1.4250
29	.0329	.2155	1.1717	2.9750	6.4640	.7808	1.3138	-.6755
30	.1581	.7781	1.4706	-3.1122	-23.5292	.7204	-.2437	-2.8821
31	1.0833	.8550	-3.0301	-7.2568	23.2711	.5611	-1.3042	.7356
32	.5931	-.8070	-1.7133	7.4237	12.3065	-.0380	-.1403	.8925
33	.3138	-1.0423	1.8724	2.7036	-25.5316	.0393	.2426	1.2332
34	-.0456	-.0243	.2902	-2.4958	13.1157	.1955	.8402	1.2614
35	.0832	.4384	1.6673	2.9890	-6.6691	1.1233	.6058	-3.3724
36	.2959	.9561	.7089	-6.4174	-17.1120	.5085	-.9288	-.9467
37	1.1762	.3066	-3.8324	-2.6918	33.3717	.2035	-.7918	1.9406
38	.4503	-.9791	-.4602	8.1122	-3.6124	-.0271	.0690	.0797
39	.1743	-.7307	1.9997	-.9141	-19.1493	0	0	0
40	-.0236	.0670	.0428	-1.6679	8.6524	0	0	0
Σ ③ × ○	-.7062	.3394	-.0208	-.273	.199	-.6959	.1210	-.0119

TABLE V.- CALCULATION OF THE INTEGRALS NEEDED FOR EXAMPLE III - Concluded

1	72	73	74	75	76	77	78
n	$\frac{1}{\omega_{15}^3}\Gamma_n(\dddot{y}_{15})$	$\frac{1}{\omega_{15}^4}\Gamma_n\left(y_{15}^{(4)}\right)$	$\Gamma_n(y_{16})$	$\frac{1}{\omega_{16}}\Gamma_n(\dot{y}_{16})$	$\frac{1}{\omega_{16}^2}\Gamma_n(\ddot{y}_{16})$	$\frac{1}{\omega_{16}^3}\Gamma_n(\dddot{y}_{16})$	$\frac{1}{\omega_{16}^4}\Gamma_n\left(y_{16}^{(4)}\right)$
0	1.7865	8.1426	-0.0335	-0.0693	0.1514	1.9516	7.1188
1	.1117	-19.8366	.2580	.8847	1.7696	-1.2097	-19.9014
2	-7.9314	2.3892	.5965	.7734	-1.6842	-6.7667	11.5107
3	5.1605	27.5582	.9844	-.9808	-2.2435	7.6917	13.8720
4	4.0571	-21.7377	.0621	-.5670	1.5329	-.0022	-18.6294
5	-2.7607	4.8180	.0154	0	.9478	0	12.0588
6	3.5401	.5843	.0621	.5670	1.5329	.0022	-18.6294
7	-7.3382	-18.6211	.9844	.9808	-2.2435	-7.6917	13.8720
8	-2.3484	27.5345	.5965	-.7734	-1.6842	6.7667	11.5107
9	9.3367	-4.4593	.2580	-.8847	1.7696	1.2097	-19.9014
10	-2.5377	-12.4980	-.0671	0	.3027	0	14.2374
11	1.1638	13.3047	.2580	.8847	1.7696	-1.2097	-19.9014
12	-.3585	-21.3858	.5965	.7734	-1.6842	-6.7667	11.5107
13	-8.6181	13.1777	.9844	-.9808	-2.2435	7.6917	13.8720
14	6.2166	17.8633	.0621	-.5670	1.5329	-.0022	-18.6294
15	3.6582	-24.2143	.0154	0	.9478	0	12.0588
16	-2.4454	11.8676	.0621	.5670	1.5329	.0022	-18.6294
17	2.4995	-8.2646	.9844	.9808	-2.2435	-7.6917	13.8720
18	-7.0252	-12.7600	.5965	-.7734	-1.6842	6.7667	11.5107
19	0	33.0746	.2580	-.8847	1.7696	1.2097	-19.9014
20	7.0252	-12.7600	-.0671	0	.3027	0	14.2374
21	-2.4995	-8.2646	.2580	.8847	1.7696	-1.2097	-19.9014
22	2.4454	11.8676	.5965	.7734	-1.6842	-6.7667	11.5107
23	-3.6582	-24.2143	.9844	-.9808	-2.2435	7.6917	13.8720
24	-6.2166	17.8633	.0621	-.5670	1.5329	-.0022	-18.6294
25	8.6181	13.1777	.0154	0	.9478	0	12.0588
26	.3585	-21.3858	.0621	.5670	1.5329	.0022	-18.6294
27	-1.1638	13.3047	.9844	.9808	-2.2435	-7.6917	13.8720
28	2.5377	-12.4980	.5965	-.7734	-1.6842	6.7667	11.5107
29	-9.3367	-4.4593	.2580	-.8847	1.7696	1.2097	-19.9014
30	2.3484	27.5345	-.0671	0	.3027	0	14.2374
31	7.3382	-18.6211	.2580	.8847	1.7696	-1.2097	-19.9014
32	-3.5401	.5843	.5965	.7734	-1.6842	-6.7667	11.5107
33	2.7607	4.8180	.9844	-.9808	-2.2435	7.6917	13.8720
34	-4.0571	-21.7377	.0621	-.5670	1.5329	-.0023	-18.6294
35	-5.1605	27.5582	.0154	0	.9478	0	12.0588
36	7.9314	2.3892	.0621	.5670	1.5329	.0022	-18.6294
37	-.1117	-19.8366	.9844	.9808	-2.2435	-7.6917	13.8720
38	-1.7865	8.1426	.5965	-.7734	-1.6842	6.7667	11.5107
39	0	0	.2580	-.8847	1.7696	1.2097	-19.9014
40	0	0	-.0335	.0693	.1514	-1.9516	7.1188
Σ ③ × ○	= -.055	.169	-.709	.2724	-.0128	-.114	.235

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TABLE VI.- SUMS NEEDED FOR EVALUATION OF THE PARAMETERS OF EXAMPLE III BY LEAST SQUARES

	1	2	3	4	5	6	7	8	9	10	11	12	13
ν	ω_ν	ω_ν^2	ω_ν^3	ω_ν^4	$\frac{1}{\Delta t}\int_0^2 xy_\nu dt$	$\frac{1}{\omega_\nu}\Delta t\int_0^2 x\dot{y}_\nu dt$	$\frac{1}{\omega_\nu^2}\Delta t\int_0^2 x\ddot{y}_\nu dt$	$\frac{1}{\omega_\nu^3}\Delta t\int_0^2 xy_\nu^{(3)} dt$	$\frac{1}{\omega_\nu^4}\Delta t\int_0^2 xy_\nu^{(4)} dt$	1 × 6	2 × 7	3 × 8	4 × 9
2	1.5708	2.4674	3.8758	6.0881	1.0427	-0.6518	-11.0678	-10.961	109.208	-1.0238	-27.3087	-42.483	664.869
3	2.4166	5.8400	14.1130	34.1056	-.1882	-2.9145	-5.7390	1.642	100.693	-7.0432	-33.5158	23.174	3434.195
4	3.1416	9.8696	31.0063	97.4091	-2.1316	-3.2009	6.5197	24.080	-39.379	-10.0559	64.3468	746.632	-3835.873
5	3.9270	15.4213	60.5591	237.8152	-.8385	-1.7570	-.7993	12.817	16.004	-6.8997	-12.3262	776.186	3805.994
6	4.7124	22.2066	104.6462	493.1335	-2.1764	-3.8832	5.9052	18.875	-24.220	-18.2992	131.1344	1975.197	-11943.693
7	5.5440	30.7358	170.3988	944.6889	-2.8302	2.0668	7.3470	-6.945	-24.007	11.4583	225.8159	-1183.420	-22679.146
8	6.2832	39.4784	248.0502	1558.5454	-.4323	1.7024	-1.1109	-4.687	4.474	10.6965	-43.8566	-1162.611	6972.932
9	6.9813	48.7388	340.2609	2375.4694	-.9567	.0375	.4663	.116	-.193	.2618	22.7269	39.470	-458.466
10	7.8540	61.6850	484.4731	3805.0426	-.6614	.6929	-.1993	-1.181	.906	5.4420	-12.2938	-572.163	3447.369
11	8.7266	76.1544	664.5721	5799.4858	-.9016	.0418	.2848	.052	-.084	.3648	21.6888	34.558	-487.157
12	9.4248	88.8264	837.1695	7890.1363	-.6965	.4501	-.0592	-.516	.348	4.2421	-5.2585	-431.979	2745.767
13	10.4720	109.6623	1148.3806	12025.8137	-.8736	.0393	.1926	.027	-.045	.4115	21.1210	31.006	-541.162
14	10.9956	120.9027	1329.3941	14617.4517	-.7062	.3394	-.0208	-.273	.199	3.7319	-2.5148	-362.925	2908.873
15	11.5743	133.9642	1550.5399	17946.3959	-.6959	.1210	-.0119	-.055	.169	1.4005	-1.5942	-85.280	3032.941
16	12.5664	157.9137	1984.4017	24936.7272	-.7093	.2724	-.0128	-.114	.235	3.4231	-2.0213	-226.222	5860.131

$\Sigma (5)^2 = 24.203509$

$\Sigma (5) \times (10) = 16.722504$

$\Sigma (5) \times (11) = -1098.1882$

$\Sigma (5) \times (12) = -1603.1935$

$\Sigma (5) \times (13) = 81036.350$

$\Sigma (10)^2 = 855.53811$

$\Sigma (10) \times (11) = -664.49463$

$\Sigma (10) \times (12) = -82282.332$

$\Sigma (10) \times (13) = 85777.101$

$\Sigma (11)^2 = 77898.726$

$\Sigma (11) \times (12) = 94741.752$

$\Sigma (11) \times (13) = -7533576.5$

$\Sigma (12)^2 = 8523669.8$

$\Sigma (12) \times (13) = -10567810$

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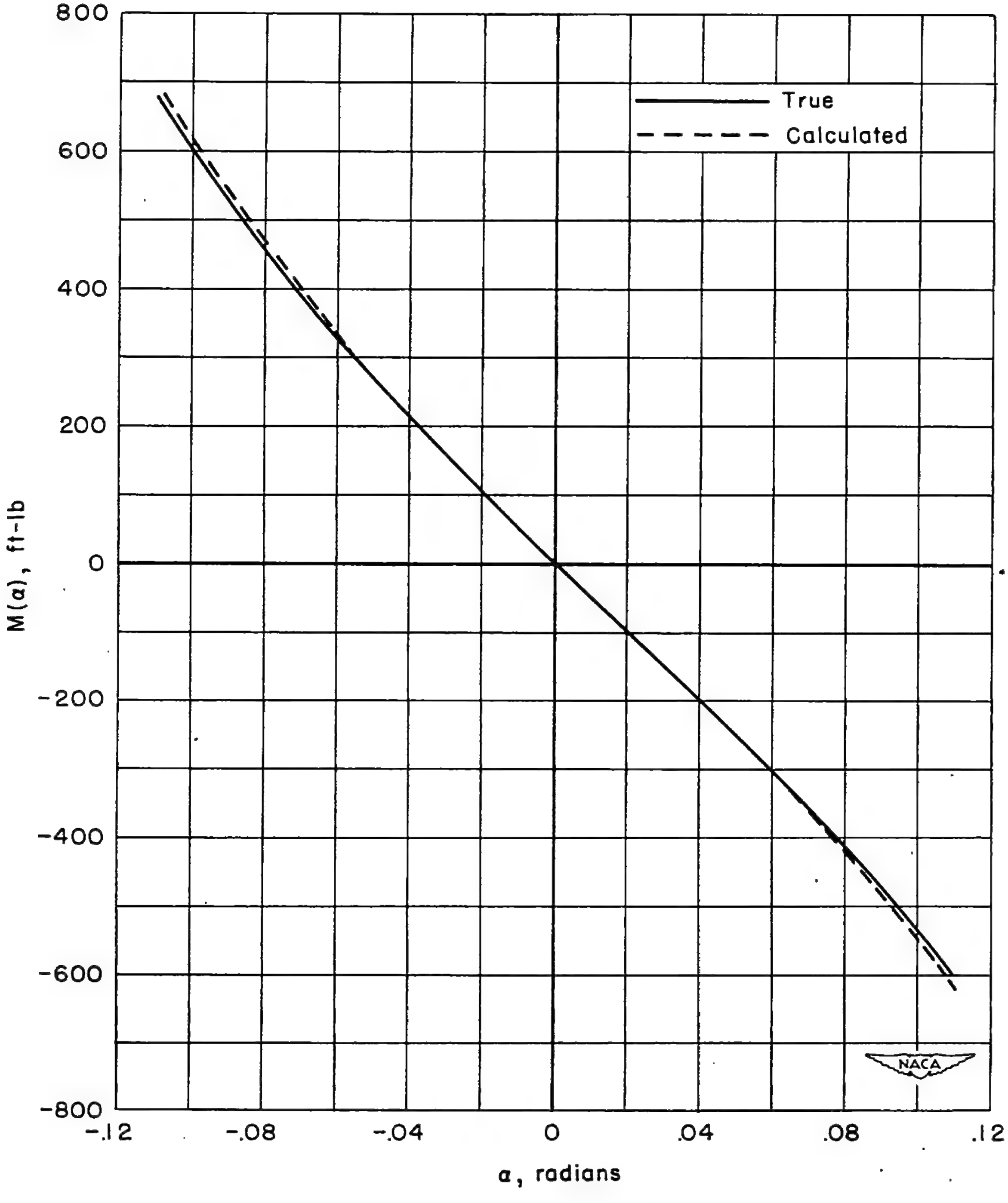

Figure 1.- The nonlinear pitching-moment curve for example I.

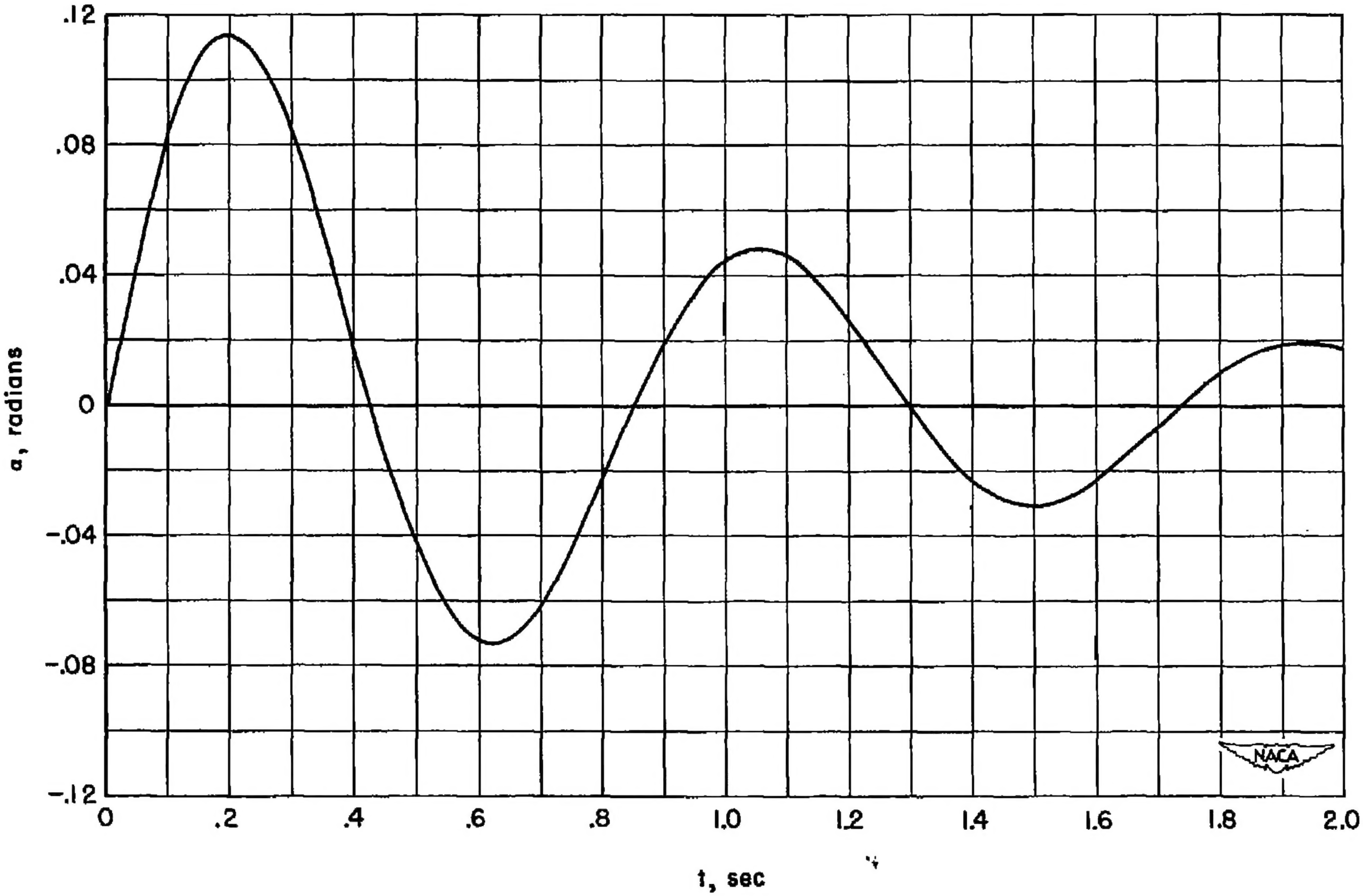

Figure 2.- "Test data" for example I.

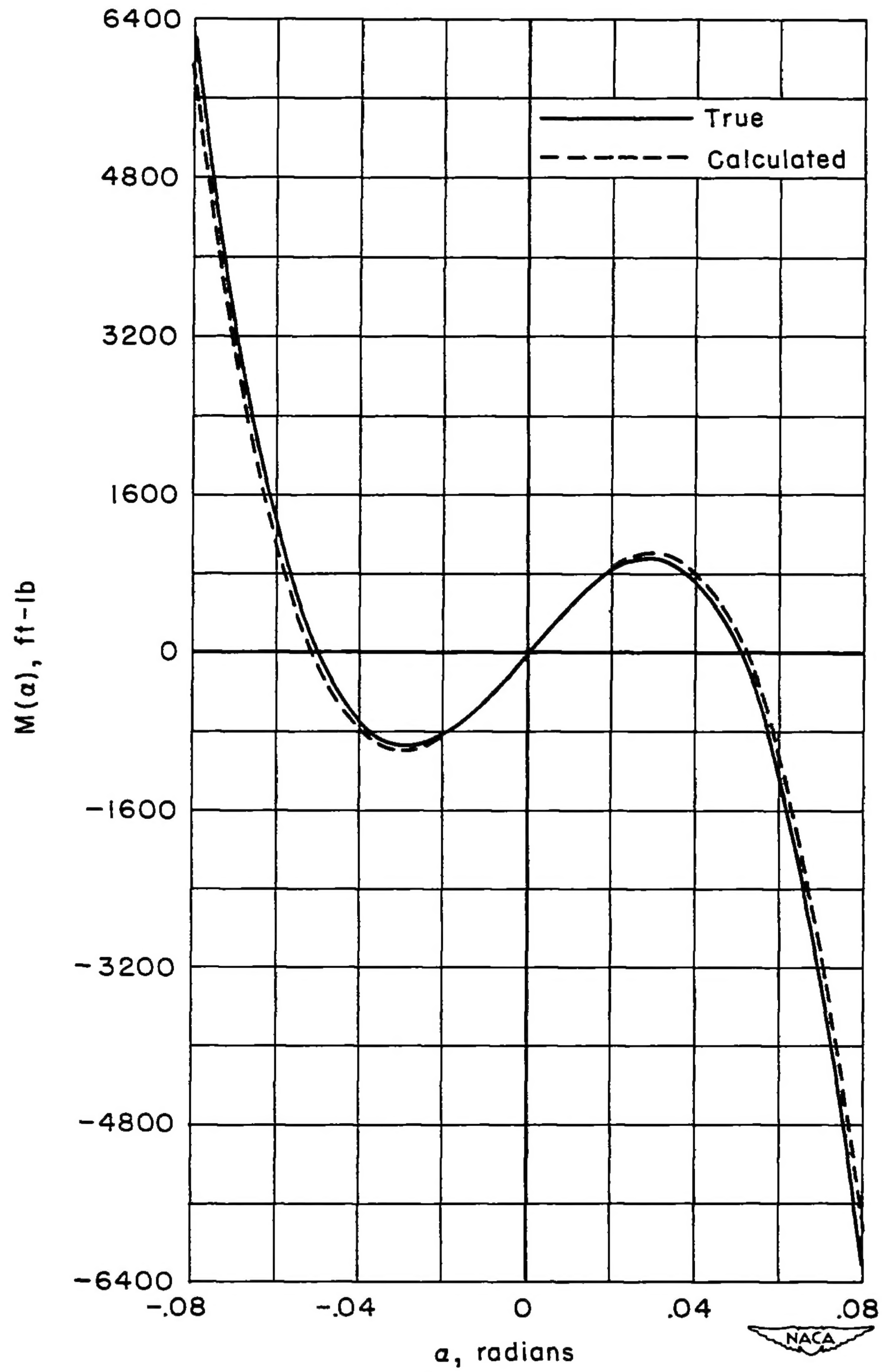

Figure 3.- The nonlinear pitching-moment curve for example II.

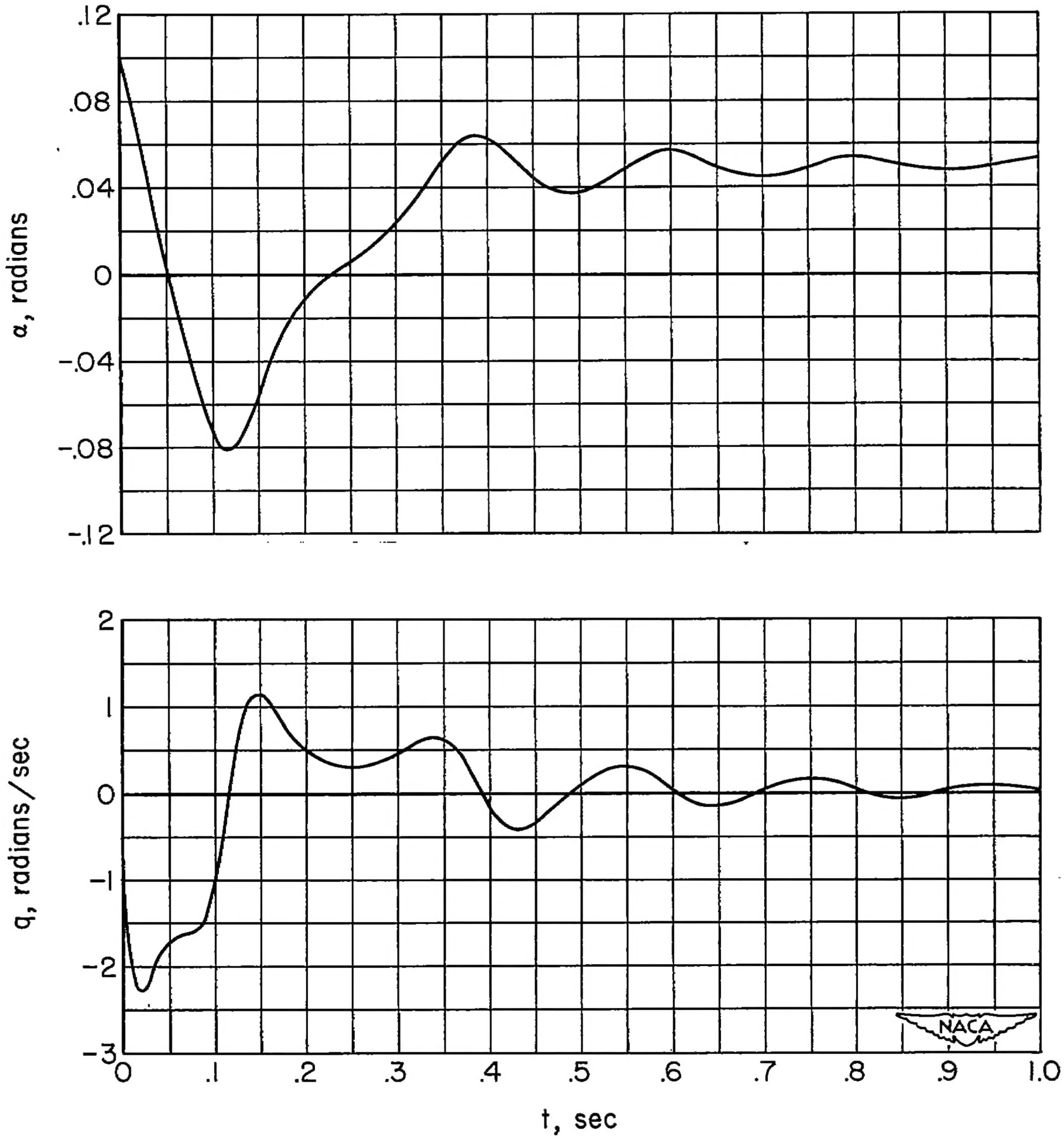

Figure 4.- "Test data" for example II.

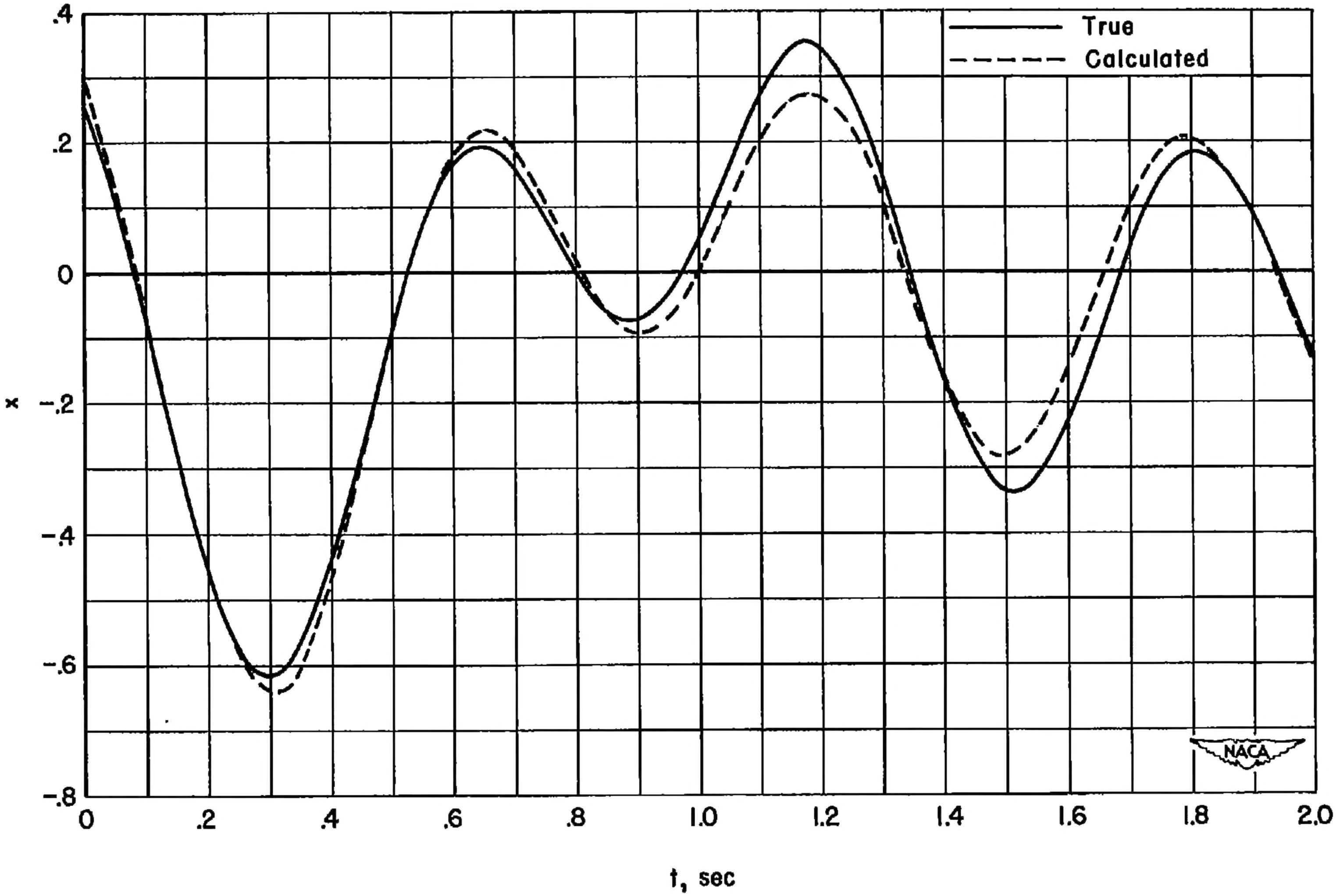

Figure 5.- "Test data" for example III.

NACA-Langley - 12-1-54 - 1000